www.ijemst.net

Mathematical Argumentation Performance of Sixth-Graders in a Chinese Rural Class

Da Zhou
Northeast Normal University, China

Jinqing Liu
Indiana University, Unites States

Jian Liu
Beijing Normal University, China

To cite this article:

Zhou, D., Liu, J., & Liu, J. (2021). Mathematical argumentation performance of sixth-graders in a Chinese rural class. *International Journal of Education in Mathematics, Science, and Technology (IJEMST)*, *9*(2), 213-235. https://doi.org/10.46328/ijemst.1177

International Journal of Education in Mathematics, Science, and Technology (IJEMST) affiliated with
International Society for Technology, Education, and Science (ISTES): www.istes.org

International Journal of Education in Mathematics, Science and Technology

2021, Vol. 9, No. 2, 213-235 https://doi.org/10.46328/ijemst.1177

Mathematical Argumentation Performance of Sixth-Graders in a Chinese Rural Class

Da Zhou, Jinqing Liu, Jian Liu

Article Info

Article History

Received:
17 July 2020
Accepted:
09 December 2020

Keywords

Mathematical argumentation
Teachers' potential influence
Sixth graders
A Chinese rural class

Abstract

Researchers have established that solid argumentation is essential for developing, establishing and communicating mathematical knowledge, which attracted substantial attention from researchers, but few have simultaneously investigated the argumentation performance of sixth-graders and their teacher's potential influence in Chinese rural classrooms. In this pilot study, 33 sixth graders in a Chinese rural class were examined, and the math teacher who had been teaching them for three years was interviewed. Findings related to the students' performance revealed the need to improve their argumentation competency, including using more diverse modes of arguments and argument representation as well as developing more advanced types of arguments (e.g., deductive argumentation). The interview finding with the math teacher indicated that the teacher's perception and knowledge might impact students' learning opportunities to conduct argumentation and, therefore, may influence students' argumentative performance. Implications and limitations of this study is discussed at the end.

Introduction

Argumentation is fundamental for deep learning in mathematics and plays an essential role in fostering students' conceptual understanding (Hanna & De Villiers, 2012; Nickel, 2019; Kanellos, Nardi, & Biza, 2013; Krummheuer, 2007). Being able to conduct a solid argumentation is essential for developing, establishing, and communicating mathematical knowledge (Lin, 2018; Stylianides, 2007; Stylianides, 2019). As such, researchers and policymakers have paid substantial attention to exploring the nature of students' mathematical argumentation and ways to help them develop argumentation skills (Lin & Tsai, 2012; Krummheuer, 2000, 2007; Stylianides, 2016). In many countries' current curriculum standards, argumentation is emphasized heavier as an important component of school mathematics (e.g., Common Core State Standards for Mathematics, 2010; Hanna & De Villiers, 2012; Healy & Hoyles, 2000; National Mathematics Curriculum in England, 2013). In the common core state standards for mathematics in the U.S.A., for example, one of the eight mathematic practices for K-12 students is focused on arguments that: "Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments…" (Common Core State

Standards for Mathematics, 2010). Chinese math curriculum also emphasizes more on argumentation. Although the concept of argumentation is not explicitly stated in the Chinese mathematics curriculum standards, some key elements in the standards target the idea of argumentation. For example, one of the four core aims that the full-time compulsory education mathematics curriculum standards (2011 version) sets for Grade 1-9 students is about arguments: "[Students should] make observations, conjectures, experiments, [and] justifications in the practice of mathematics …" (Ministry of Education of the People's Republic of China, 2011).

However, it is not easy to implement argumentation in the math classroom, especially in rural China. Previous studies indicate that experienced teachers in China who have taught for more than a dozen years often pay primary attention to students' acquisition of mathematical knowledge while ignoring the cultivation of their argumentation ability (Hu, Wang & Xu, 2013; Liu, 2013). This phenomenon is particularly prevalent in rural areas. Through empirical surveys, Chang (2005) and Lu, Jiang & Li (2011) found that rural math teachers are reluctant to accept the new orientation of the new curriculum and still prefer to use teacher-centered methods, which leads to a situation that students rarely have opportunities to participate in the mathematical discussion, which also limited their opportunities to argue for their mathematical ideas with each other. Such teaching may negatively affect students' argumentation performance. To gain a better understanding of how Chinese students in rural areas may perform on argumentative tasks and their teachers' role in their argumentation development, this pilot study focused on exploring a particular case: a sixth-grade classroom taught by an experienced math teacher for three years (grade 4-6) in rural China. We surveyed the whole classroom and interviewed their math teacher to collect data. The goals of the study were to examine these students' argumentation skills and gain a preliminary understanding of how their teacher's self-reported knowledge perspective and practices regarding argumentation may be related to students' argumentation performance.

Literature Review

Defining the Concept of Argumentation

Researchers interested in the teaching and learning of argumentation in school mathematics have long sought to accurately define the concept of argumentation, but definitions have differed according to different research purposes. Generally, these definitions can be grouped into two major perspectives: argumentation as a process of social negation or debate (social perspective), and argumentation as a cognitive process of providing supportive warrants and evidence for claims (cognitive perspective). Some researchers (Hunter, 2007; Kosko, Rougee & Herbst, 2014; Krummheuer, 2007; Lin, 2018; Martino & Maher, 1999; Stylianides, 2007 et al.) have tended to focus on the social perspective on argumentation and explore its social interaction aspects. Krummheuer (2007) recommended that mathematical argumentation should be an instructional goal, and teachers should create classroom environments conducive to exchanging ideas and debates in which students have ample opportunities to justify their mathematical thinking to develop rigor in their mathematical reasoning. In addition, Hunter (2007), Kosko et al. (2014), and Martino & Maher (1999) conducted an empirical survey on how teaching questioning played roles in the classroom's collective argumentation. Similarly, both Lin (2018) and Stylianides (2007) viewed mathematical argumentation as a social interaction process in which students increase their understanding by communicating their ideas in an effort to convince each other.

A number of other researchers have taken a more cognitive perspective on argumentations (Osborne, Henderson, MacPherson, Szu, Wild, & Yao, 2016 et al.), focusing on students' construction of arguments, such as identifying warrants and providing evidence for claims. Osborne et al. (2016) indicated that argumentation required the ability to construct and evaluate evidence and critique by examining the warrant connecting the evidence and a claim. The construction of knowledge involves the process of using arguments to support a proposition or hypothesis, whereas critique is the process of identifying the strengths and/or weaknesses of an argument (Osborne et al., 2016). Toulmin (1958) constructed key elements in argument structure, beginning with a claim that is a "conclusion we are seeking to establish" (p. 90), which is supported by data as evidence. The relation between the claim and the evidence is provided by a warrant, which is reasonable justification, such as a mathematical axiom or a theorem.

Combining the two perspectives, it can be said that in math education, students' argumentation ability should be developed cognitively and supported socially. In this study, we focus on examining students' cognitive development of argumentation, defined here as the ability to utilize mathematical knowledge to employ evidence and warrants to construct, critique, or support claims. Furthermore, we also consider the potential influence of the math teacher's knowledge, perspective, and practice regarding argumentation on students' argumentation performance. Because math teachers are the ones who largely affect the social dynamics of math lessons and determine the discussion or argumentation opportunities students may receive in math classrooms.

Studies on Students' Mathematical Argumentation

The relevant literature suggests the prevalence of case studies of individuals or small groups to investigate students' performances of mathematical argumentation (e.g., Krummehuer, 2000; Stylianides, 2016; Van Ness & Maher, 2019 et al.). From these cases, we gained some understanding of the range of what students are able to do in conducting argumentation. For example, Krummehuer (2000) used math tasks to trigger students' engagement and investigated the student-teacher and student-student dialogues, and found that argumentation was the key process by which students learned mathematics and advocated that students should receive plentiful opportunities of engaging in collective argumentation in mathematics classes. Van Ness and Maher (2019) used Toulmin-style diagrams to analyze a 90-minute video of a group of fourth-graders discussing fraction comparisons and found that nine-year-olds were able to share their ideas, consider the input of others, and make revisions based on evidence. Meanwhile, they also found that students experienced difficulties informally expressing their ideas in full sentences.

From reviewing the above case studies, we found that although case studies focused on individuals or small groups of students offered realistic descriptions and insight into particular students' oral argumentation, they don't provide a comprehensive picture of the diversity of the whole students' performances in a natural classroom. Thus, from a cognitive perspective, we proposed that using a class as an analytical unit to examine students' written work, which would include the reasoning of students who disinclined or felt uncomfortable to participate in oral argumentations, could offer a slightly different picture of a wider range of students' argumentation abilities.

Another feature of studies of students' argumentation performance is that they tend to focus on particular grade levels. For example, Stylianides (2007) analyzed mathematics teaching episodes in a third-grade class to examine the major features of students' arguments to determine whether they provided evidence in support of teaching argumentation at an early elementary level. Lin (2018) explored the argumentation of 24 third and fourth graders and recommended conjecture activities to improve their argumentation abilities. However, few studies focusing on sixth graders in the stage of primary school.

Align with the trend of focusing on particular grades, and this study focuses on sixth graders. In China, the sixth grade is an important juncture between elementary and middle school, and argumentation in the math curriculum and textbook changes from inductive to deductive arguments with formal proof from elementary school to middle school (Ma & Cao, 2017), which is a cognitive jump for many students. Thus, preparing sixth-graders' argumentation at a certain level is essential to support their further mathematical development in middle school. Therefore, assessments of sixth-graders' argumentation performance are needed not only to support secondary school teachers as to how to design mathematical teaching on students' prior knowledge but also to inform primary teachers as to how to prepare their students for the transition.

To explore or assess students' mathematical argumentation, task design is a very important approach. Because what students are asked to do in the task determines the information researchers may gain and informs the possible direction of improving teaching argumentation teaching (Watson & Ohtani, 2015). For example, Stylianides (2016) designed tasks to reveal students' ability to reason and construct arguments. To understand the quality of students' argumentation, Lin (2018) surveyed with math tasks related to conjecturing. Although both researchers discussed how they developed argumentative tasks, Stylianides (2016) developed a range of criteria for tasks that included creating conflict or uncertainty for students, giving rise to a statement that needs to be proved, and allowing for multiple solutions for which students can draw on different mathematical tools. These studies indicated that task design should be connected with some essential principles. To summarize, in a review of studies on students' mathematical argumentation, we identified an important research call for examining the argumentation performance of sixth-graders to gain a more holistic map of students' diversity argumentation skills in a natural classroom when they are in elementary school up to the transition to middle school.

The Role of Teachers in the Teaching of Argumentation

Teachers' knowledge plays a critical role in supporting students' mathematic learning (Ellis, Özgür, & Reiten, 2018). Evidence shows that teachers' mathematical content knowledge has an important effect on students' learning of argumentation (Livy, Herbert, & Vale, 2019; Stylianides & Ball, 2008). Livy, Herbert, and Vale (2019) illustrated that mathematic teachers with rich mathematical content knowledge could differentiate between mathematical ideas, select and use efficient argumentation strategies when teaching, deal with students' misconceptions, and use more formal mathematical argumentation language during teaching. Stylianides and Ball (2008) argued that mathematics teachers' understanding and ability to distinguish modes of argumentation (e.g., empirical arguments or deductive arguments) was a crucial aspect of mathematical content knowledge. As

Martin and Harel (1989) noted, if elementary teachers led their students to believe that empirical arguments were formal proof, then the idea of mathematical proof in middle school geometry and other courses would be difficult for their students to capture. Meanwhile, knowing different ways of presenting or forming argumentation is also important. Stylianides and Ball (2008) further argued that mathematics teachers' understanding and use of accepted mathematical language in proofs were also key aspects of mathematical content knowledge and important factors influencing how students form and represent mathematical ideas were formed. Also, findings from the existing research show that teachers' pedagogical knowledge is essential for the teaching of argumentation (e.g., Ball, Thames, & Phelps, 2008; Conner, Singletary, Smith, Wagner, & Francisco, 2014; Corleis, Schwarz, Kaiser, & Leung, 2008; Ellis, Özgür, & Reiten, 2018; Silverman & Thompson, 2008 et al.). Ellis and her colleagues (2018) showed that mathematics teachers are required to choose appropriate tasks, determine when and how to foster students' thinking, and decide when to allow students to engage in a productive struggle during instruction, all of which are important for the development of students' reasoning and argumentation.

While these studies have confirmed the importance of mathematical content and pedagogical knowledge for the teaching of argumentation (Silverman & Thompson, 2008; Stylianides & Ball, 2008) for high-quality argumentation instruction. Meanwhile, how the teacher perceives the importance of argumentation also influences how much emphasis teachers put on teaching argumentation. We consider teachers' perception of argumentation may also influence students' argumentation learning. As such, further exploration is needed for a better understanding of how teachers' knowledge, perspective, and practice of argumentation relate to students' argumentation performance. Thus, this study will concentrate on an experienced (20+years) math teacher in a rural school in China to investigate his perceptions of argumentation and teaching of argumentation, thereby find some potential valuable problems about teacher's argumentation teaching to extend the current study.

An Analytical Framework of This Study

For the analytical framework of this study, we drew on Stylianides' (2007) definition of proof as "a mathematical argument, a connected sequence of assertions for or against a mathematical claim" (pp. 291) and his three main components of mathematical argumentation: (1) a set of true and available statements requiring no further justification, (2) valid and known modes of argumentation, and (3) appropriate and known modes of argument representation. Referring to Dewey's (1902) contention that teachers should focus on the experience of children when they learned knowledge instead of the knowledge itself. Stylianides (2007) focused on students as mathematical learners rather than mathematics as a discipline. Thus, the components of mathematical argument were rephrased as (1) a set of available statements in the classroom community, (2) known modes of arguments in the classroom community, and (3) known modes of argument representation in the classroom community, in which the community was principally a primary classroom. Stylianides classified these components based on primary school students. Subsequent researchers have conducted studies of primary students' argumentation based on these components as an analytical framework with informative results (Lin & Tsai, 2012; Lin, 2018; Stylianides, 2007). However, in China, researchers have not used this analytical framework to understand students' performance in mathematical argumentation, hindering cross-national

comparisons that be used to promote educational development and making a case for a universal framework used in all countries.

Balacheff (1988) distinguished four levels of proof. Level 1, Naïve empiricism, involves testing a mathematical statement by verifying a few cases. Level 2, The crucial experiment, refers to using a special or extreme case to show the truth. Students at this level realize the problem of generalization and the insufficiency of basing verification merely on few examples. Level 3, The generic example, refers to a mathematical statement determined to be true after showing a representative operation. Level 4, The thought experiment, involves investigating the properties of an operation to justify the truth of a mathematical statement. The thought experiment can be seen as eliminating the particular in the process of proofs relying on a generic example. Stylianides (2019) considered naïve empiricism and the crucial experiment as weak argumentation, and the generic example and the thought experiment as strong argumentation.

Stylianides (2007) also described the modes of argument, such as inductive arguments, deductive arguments, and the construction of counterexamples. Lin and Tsai (2012) analyzed fifth graders' mathematical proofs and found their modes of argument involved the use of counterexamples and empirical induction from finite cases, and argument representations included textual, pictorial, and symbolic descriptions, as well as descriptions combining text, pictures and/or symbols. This analytic framework (Table 1) is useful for mapping sixth-graders' written argumentations in three aspects and allows for comparing the findings of this research with those of other studies.

Table 1. An Analytical Framework of Students' Argumentation in this Study

Components of mathematical argument	Types
A set of available or accepted statements	✧ Naïve empiricism ✧ The crucial experiment ✧ The generic example ✧ The thought experiment
Known modes of arguments	✧ Inductive arguments (finite cases) ✧ Deductive arguments ✧ Construction of counterexamples
Known modes of argument representation	✧ Textual description ✧ Pictorial description ✧ Symbolic description ✧ Mixed description

The Current Study

Contributing to bridge the above-identified research gaps, the purpose of this is to examine sixth-graders' argumentation performances in a natural classroom at a rural elementary school in China and to understand how

their math teacher's perception and teaching on argumentation combining with their argumentation performance. This pilot study will provide us a better understanding of how Chinese rural students may perform in solving an argumentative task as well as the teacher's potential role in facilitating their argumentation development. The specific research questions are as follows: (1) How do a class of sixth-graders in a Chinese rural area perform in solving an argumentative task? (2) To what extent does the experienced teacher's self-reported knowledge, practice, and perspective align with his students' argumentation performance?

Methods

To answer the first research question, that is, how a class of sixth graders in rural China perform in solving an argumentative task, we first designed an argumentative task and then identified a class in a rural school to participate in this study. We invited all the students in the class to solve the task in a paper and pencil format and coded their responses to determine their performance. To answer the second question, that is, what important factors influence students' argumentation performance may be from the experienced teacher's practice of teaching, we conducted a half-hour, semi-structured interview with the teacher who had been teaching these students for three years. The interview questions focused on understanding the level of the teacher's argumentation knowledge and how he viewed the students' argumentation performance in the argumentative task.

Participants and Context

The participants in this study were 33 sixth graders (15 boys and 18 girls) who constituted an entire class in a rural primary school and their math teacher, who had been teaching them in the fourth through sixth grades (all participants' names used in this article are pseudonyms). The average age of the students was approximately 12 years. Their average socioeconomic status, as defined by the Chinese Ministry of Education, was low. Most of these students' parents were farmers, who typically paid less attention to and had a lower impact on their children's formal schooling than parents in other employment sectors (From the interviewed teacher's description). As such, it looks like that these students' daily school education, especially the teacher's daily instruction, might be the major if not the sole educational input for these students' mathematical argumentation development, especially under the context that they have been taught by the math teacher for three academic years.

The rural school in this study, which was located in a town in northeast China, was chosen because it was a typical rural school and willing to participate in this study. We purposefully chose an experienced teacher, Mr. Chen, (pseudonym), from this school. Mr. Chen had worked in this school for 29 years, which definitely meets the criteria of an experienced teacher. He held a vocational diploma, which is normal for experienced teachers who have taught more than two decades in China's rural elementary schools. He was recruited because he expressed an interest in participating in this study. He shared some time of his math classroom for us to conduct the students' survey and completed a follow-up interview with us voluntarily.

Task Design

For this study, the following mathematical task was designed based on the fifth-grade mathematic content to examine students' understanding of the rectangular area formula:

> When teaching a lesson on the perimeter and the area of a rectangle. Mrs. Wang proposes the idea that: "For any rectangle, if the length is halved, the width is doubled, then the area of the rectangle will always remain the same." Do you agree with this idea? Please explain the reasons for your judgment (English version of this task).

Figure 1. The Argumentative Geometrical Task

To develop this task, we considered the following mathematical task design principles:

(1) The task should elicit students' mathematical argumentation by providing the mathematical claim, following which students should use known modes of arguments to justify whether or not the mathematical claim could be supported by the evidence they have found. For example, in the above task, the claim is that "For any rectangle, if the length is halved, the width is doubled, the area of the rectangle will always remain the same." Students are asked to justify their agreement or disagreement with this claim.

(2) The mathematical task should be based on the Full-time Compulsory Education Mathematics Curriculum Standards, which were the core guidelines for mathematical teaching in China, followed in almost all mathematical teaching. In the above task, the topic of perimeter and area of the rectangle is an important lesson for elementary students. In the process of solving the mathematical problem, students are cognitively challenged to apply their knowledge of the rectangle area to support their judgments, which would help them develop their mathematics understanding and argumentation ability.

(3) The mathematical task should require knowledge already acquired by the students. That is to say, and their prior knowledge can provide a foundation for solving the problem. In this case, the relevant content had been taught in the fifth grade. As we were developing the geometrical mathematic task for this study, we obtained advice from two experts majoring in mathematics education, which contributed to establishing a reasonable level of content validity.

Procedures

Collecting Students' Written Responses

In the spring semester of 2019, the teacher administered the survey during his math teaching time, and all students in the class completed the argumentative task using pen-and-paper, taking as much time as they needed, which was roughly ten minutes. The teacher closely monitored the survey, and students solved the task independently, ensuring the reliability of data collection. All students completed the task voluntarily and anonymously. All names in this reporting are fictitious. Their responses were collected by the researchers and not shown to the teacher as the results may influence Mr. Chen's response in the follow-up interview.

Conducting Teacher's Interview

After collecting the written responses, the first author cooperated with a math teacher (for reasons that will be explained in the results) to code students' responses. The coding results enabled us to identify three major characteristics of students' argumentations: the majority used induction rather than deduction methods, verbal justification dominated, and few students offered counterexamples. Based on these observations, the research team designed a protocol for the semi-structured interview that focused on three aspects: Perspective on argumentation (e.g., is it important?); knowledge of argumentation (e.g., what is the deductive method?), and understanding of students' argumentation (e.g., why do your students use verbal justification more?). Guided by the interview protocol, we conducted a face-to-face interview. In addition, before the interview, the first author explained the purpose of the interview to alleviate the teacher's nervousness and, ensuring his engagement. The interview was audio-recorded and transcribed into Chinese for later analysis. Some selected statements were translated into English for reporting the findings. The accuracy of the Chinese-English translation was checked by two fluent English Chinese.

Data Analysis

Analyzing Students' Responses

To ensure coding reliability, we engaged another math teacher, Ms. Li (pseudonym), to be a second coder of the students' responses. She was selected for two reasons. The first was that she was willing to contribute her spare time to the coding voluntarily. She expressed that coding students' argumentation was also a professional development opportunity for teaching argumentation. The second reason was that she wasn't the teacher of the class. She would not have any biases, which might be a problem if the class teacher was engaged in the coding. Also, we needed to control for the effects of irrelevant factors on the math teacher's interview.

As one of the coders, the first author provided Ms. Li with the coding training. During the coding training, she was asked to solve the argumentative task independently to get familiar with the task. Then, the first author explained the scoring rubric to her in detail. In the end, both coders coded about 10 different kinds of students' responses. Once our coding was in alignment, we coded the 33 students' responses independently and then compared our codes, which matched for 28 responses, an agreement rate of 85%. We then discussed our disagreements, and we reached a consensus on all the tasks at the end. The coding process ensured score consistency reliability.

Analyzing Teacher's Interview Protocol

The interview was a conversation between Mr. Chen and the first author of this paper on the teacher's argumentation subject and pedagogy knowledge. Specifically, the interviewer asked open-ended questions such as why argumentation was important, what deductive and inductive argumentation methods were, and why students used textual representation modes more than other modes, and so on. After transcribing the audio-recorded interview into text, we regarded the sentence as the coding unit and coded the whole text, which helped

us quantify the text and identify the position of each sentence in the text. Clarifying the unit of coding text facilitated the analysis of the text. Third, we conducted a meaning analysis to relate the interview data to the research purpose. To sum up, transcribing, coding, discussing, and conducting meaning analysis facilitated our analysis of the teacher's perceptions.

Results

In this section, we will present general trends in the results, exemplify them with some students' mathematical argumentations, and offer a further explanation based on their math teacher's interview. It includes three sub-sessions: students' examples and teacher's interview findings for accepted statements, known modes of arguments, and known modes of argument representation. Table 2 displays the general trends of students' argumentation.

Table 2. Students' Argumentation in the Three Categories of Components

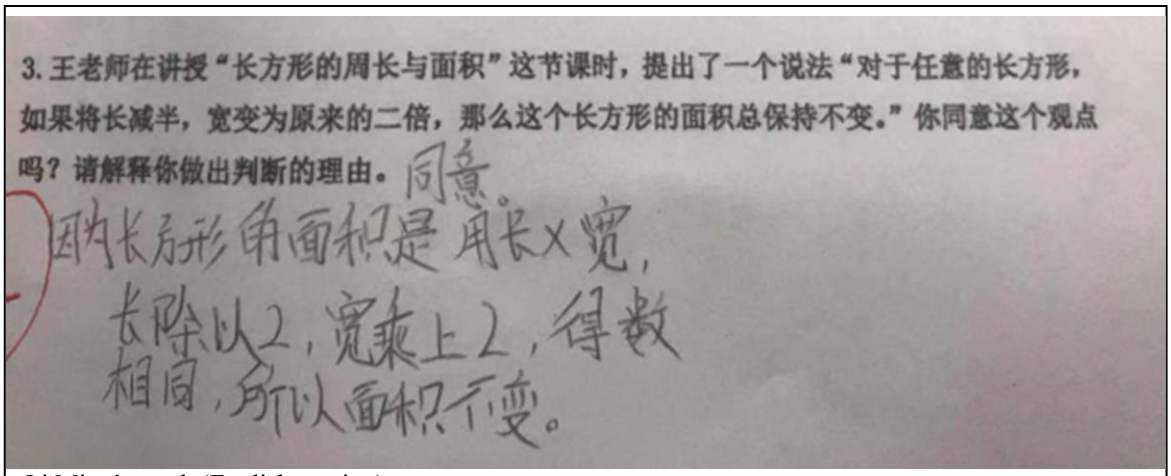
Components of mathematical argumentation	Types	Numbers and percentages
A set of available or accepted statements	• Naïve empiricism	13 (39%)
	• The crucial experiment	0
	• The generic example	4 (12%)
	• The thought experiment	1 (3%)
	• An invalid argument	15 (45%)
Known modes of arguments	• Inductive arguments	13 (39%)
	• Deductive arguments	5 (15%)
	• Counterexamples	0
	• An invalid argument	15 (45%)
Known modes of argument representation	• Textual description	18 (55%)
	• Pictorial description	0
	• Symbolic description	0
	• Mixed description	11 (33%)
	• Nonresponses	4 (12%)

Regarding a set of available or accepted statements, as shown in Table 2, 15% (N=5) of the students conducted strong mathematical argumentation, including the generic example and the thought experiment; 39% (N=13) of the students presented a relative weaker mathematical argumentation, including naïve empiricism; and 45% (N=15) of the students did not present valid arguments. Concerning known modes of argument, 54% (N=18) of the students presented genuine arguments. Specifically, 39% (N=13) of the students used the inductive method to justify their ideas, almost all using one case as evidence, and 15% (N=5) of the students used the deductive method, in most cases correctly. No students were based on counterexamples to make argumentation. As noted, the remaining 45% (N=15) of the students did not provide genuine arguments.

In terms of known modes of argument representation, 55% (N=18) of the students used textual description, 33% (N=11) combined textual description with symbolic description, and 12% (N=4) did not provide a description. To sum up, the first two of these results have a close connection. Students that used naïve empiricism and the crucial experiments to make argumentation are those students that used deductive arguments, and students that used the generic example are those students that used inductive arguments. However, results that used modes of argument representation seem to be not directly associated with the first two results.

Students' Examples and Teacher's Interview: A Set of Accepted Statements

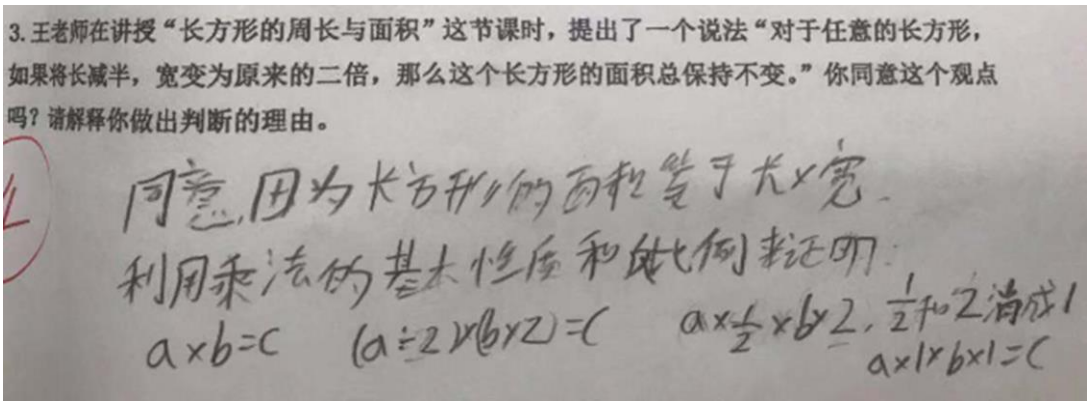
Five students showed strong argumentation, of whom four students presented a generic example, as illustrated by Li Ming's work (Figure 2a).



Li Ming's work (English version):
Agree. The formula for the area of a rectangle is length times width. If length is divided by 2, width multiplied by 2, the final result is the same, so the area is the same.

Figure 2a. Li Ming's Work Using the Generic Example

As in this example, students in this group used generic information such as the formula for the area of a rectangle, "length times width," and "the properties of multiplication" (a number times the same number and then divided by the same number results in the original number) to make an argumentation which is valid for all rectangles. Another of the five students who produced strong arguments, Zhang Hong (see Figure 2b), presented a thorough and detailed argument at the level of a thought experiment. In his example, he first stated his claim, which was that he agreed with the given proposition. He then used the term "because" to introduce his evidence, which consisted of the formula for the area of the rectangle, the property of multiplication, and knowledge of ratio. In the end, he rewrote the argument in symbolic form to show how these mathematical foundations work together: a×b=c (the formula for the area of a rectangle), (a÷2)×(b×2)=c (the property of multiplication), $a\times\frac{1}{2}\times b\times 2=a\times b\times(\frac{1}{2}\times 2)$= a×b×1 (applying the knowledge of ratio)".

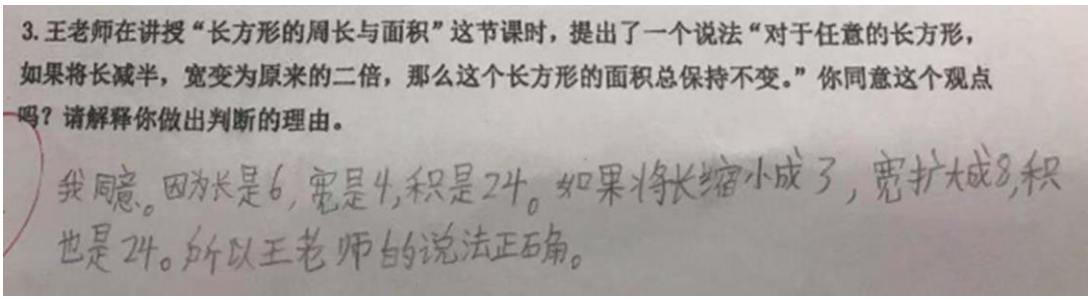


Zhang Hong's work (English version):

Agree. Because the formula for the area of a rectangle is equal to length×width, using the properties of multiplication to prove that: a×b=c, (a÷2)×(b×2)=c, $a\times\frac{1}{2}\times b\times 2$， ½ and 2 reduce to 1,a×1×b×1=c.

Figure 2b. Zhang Hong's Work Using the Thought Experiment

Zhang Hong's process of proving in this task indicated that he may have possessed a strong argumentative ability. Both the generic example and thought experiment approaches are regarded as important argumentation strategies at a high level of proving (Balacheff, 1988), but only 15% of the students in Mr. Chen's class achieved this level. Besides, 13 students presented naïve empiricism or used finite examples as evidence to make arguments (see Figure 2c), and 15 students showed invalid argumentation.



Cui Yan's work (English version):

Agree. Because the length is 6, the width is 4, so the multiple is 24. If the length were reduced to 3, the width increased to 8, the area of the rectangle would also be 24. Therefore, Mrs. Wang's statement is true.

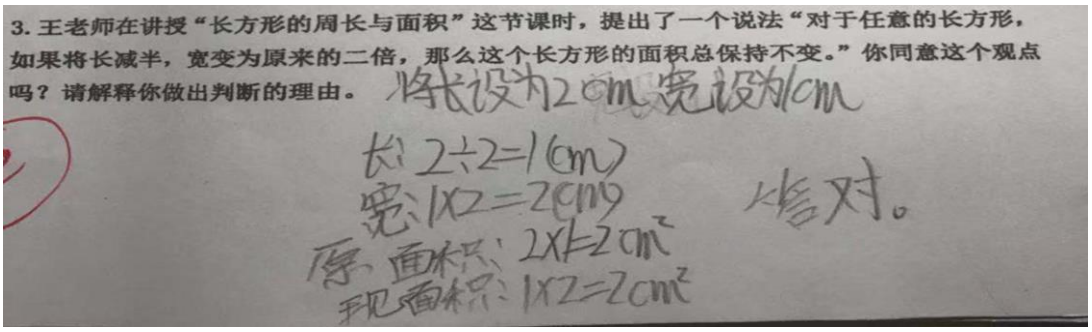
Figure 2c. Cui Yan's Work Using the Naïve Empiricism

Figure 2c shows that Cui Yan used one specific example, a 6×4 rectangle, to test the argumentation. She followed the process of halving 6 into 3 and doubled 4 to 8, and then found 6×4 is equal to 3×8 and both are equal to 24, showing that she understood Mrs. Wang's claim clearly and possessed a basic idea of argumentation. However, she did not justify the claim as applicable to all rectangles. Regarding the situation that few students achieved the highest level of argumentation and the majority preferred to use a singular case to justify their argumentation, we interviewed Mr. Chen. First, we were curious about Mr. Chen's perception of the value of teaching and learning argumentation. Because a math teacher who does not value argumentation is

unlikely to offer plentiful argumentation opportunities to students with limited classroom time. Based on the interview data, Mr. Chen highly valued argumentation. He stated that "Argumentation is very important because one claim or one theorem needs to be proved by some data or evidence. If there is no evidence, the claim you established is not true, so argumentation is very important." This statement indicates Mr. Chen perceived the importance of argumentation; however, it sounds like from a more mathematical proof perspective rather than from a pedagogical perspective, which seems to influence his instructional decisions. For example, when asked how he taught argumentation in the math classroom, he replied, "In my teaching, I asked students to have a short discussion on one claim, and then give their answers. Although I would like to give students much more time, the teaching time was limited. I have to finish teaching tasks within the required teaching time." Mr. Chen's statements indicate that he seemed to more of the "answer" rather than the argumentative process when he under time pressure. To sum up, Mr. Chen's answer indicates that one of the challenges of effectively conducting argumentations in class was the limited teaching time, which may cause students' inadequate performance on this task.

Students' Examples and Teacher's Interview: Known Modes of Arguments

The four known modes of arguments are inductive arguments, deductive arguments, counterexamples, and invalid arguments. Five students made deductive arguments, including the generic example and the thought experiment discussed, such as Li Ming's work (Figure 2a) and Zhang Hong's s work (Figure 2b). Thirteen of the students made inductive arguments, all of whom supported their claims with a single as evidence. For example, Wang Hua (see Figure 3) used a case in which the original length of a rectangle is 2cm, and the width is 1cm. His calculations showed that both the original and the altered areas were 2 $cm^2$and so he concluded that Mrs. Wang's idea is correct. Cui Yan's work (Figure 2c) mentioned before is also in this group.



Wang Hua's work (English version):

We can set the length to be 2cm, set the width to be 1cm.

Length: 2÷2=1(cm)

Width:1×2=2(cm)

Original area: 2×1=2cm^2

Current area: 1×2=2cm^2

Answer: Yes

Figure 3. Wang Hua's Work Using an Inductive Argument

That many more students used inductive rather than deductive argumentation may be due to the high-cognitive demand for conducting deductive arguments. However, in some international studies, it has been found that elementary students are capable of using informal language to present deductive processes (Lin, 2018; Stylianides, 2016; Van Ness & Maher, 2019; Yore, Pimm, & Tuan, 2007). When sixth-graders enter middle school in China, they are required to master formal deductive arguments. Thus, they are expected to master informal deductive argumentation in primary school.

To better understand why more students in solving this task used inductive argumentation, our interview with the math teacher probed his methods of argumentation and found that he relied on giving examples or presenting data as his argumentation mode. He shared his understanding that "inductive argument is to show some data to students, ask students to observe the data, then induce a theorem or a definition." However, the math teacher could not define the deductive argument. The interviewer then explained the meaning of a deductive argument and asked for his opinion on the importance of developing students' deductive argument abilities during elementary school. The teacher expressed a relatively strong opinion that learning deductive argument is not suitable for elementary students, stating, "I think it is not suitable. Students should focus on inductive arguments during elementary school because their cognitive levels are limited." Overall, based on the interview data, we found that the students' use of the argumentation modes seemed to be aligned with their teacher's general knowledge and perceptions of these modes. Possible, because Mr. Chen modeled inductive argumentation as the mode in class, and he expected his students to use inductive argumentation, his students tended to use single examples to support their claims.

Students' Examples and Teacher's Interview: Known Modes of Arguments Representation

Eighteen students used textual description to make arguments, as illustrated by Zhou tian's work (Figure 4a), and eleven students used combined textual and symbolic descriptions to make arguments, as in Du Feng's work (Figure 4b).

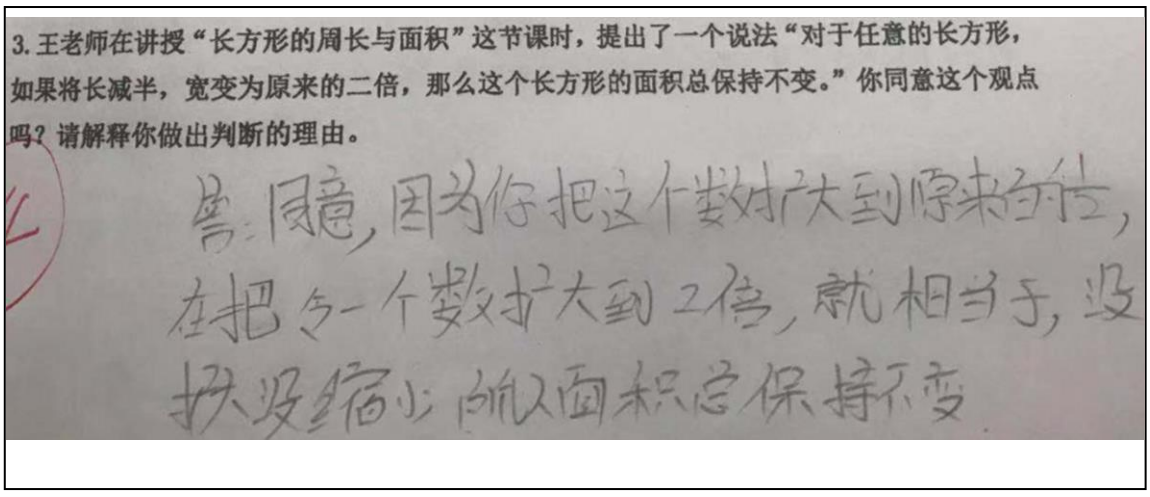


Figure 4a. Zhou Tian's Work Using a Textual Description

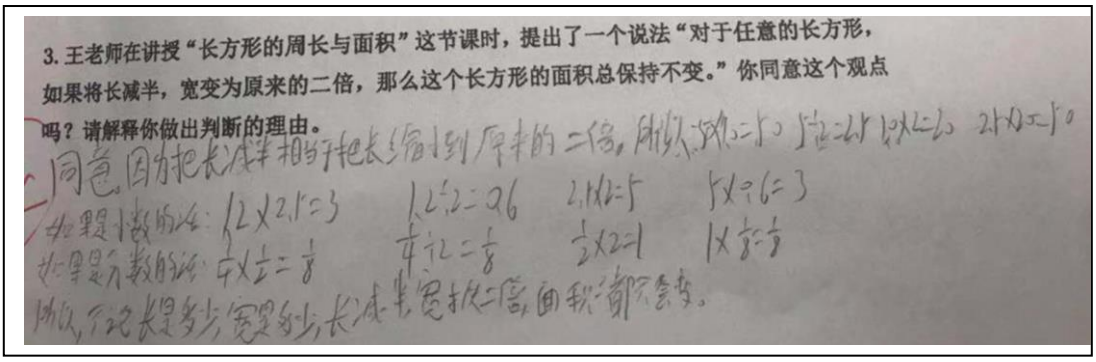


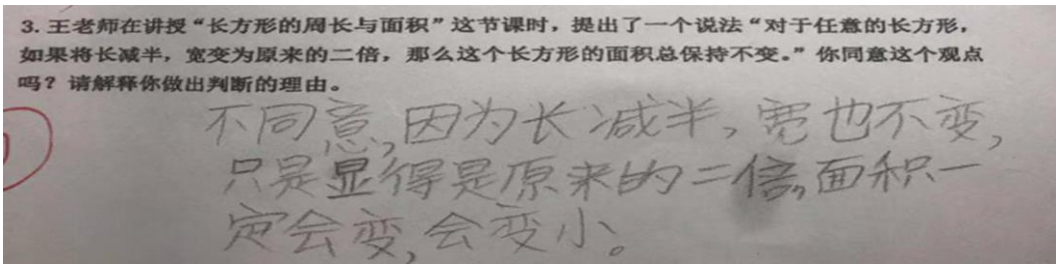
Figure 4b. Du Feng's Work Using Textual and Symbolic Description

No students used pictorial representations, which was counter to our expectation that elementary students would prefer to use visual representations, a more intuitive way, for problem solving. Again, in our interviewer with the math teacher, we sought an explanation for students' preferences for textual or symbolic rather than pictorial representations. First, we asked him if he understood the meaning of representation, and when he said he did not, the interviewer explained the definition of representation and then asked for his perception of the importance of representation in math learning. The teacher responded: "I think it is quite important because representation can improve students' abilities to make inductive arguments. For example, if students can use mathematical symbols to represent general rules, they have reached a certain level of cognition. In general, students did not use formal language to explain their claims". The above statement suggested that the teacher valued representation in some way. Therefore, the interviewer asked how he encouraged his students to represent a problem in multiple ways. He said, "I encourage students to solve a problem in multiple ways. For example, by drawing a line graph or a circle or by showing one group of data to solve a problem or to justify one claim." His above statement indicated that he tended to see different representations more as different solution methods, but he also mentioned some visual representations. When asked, "What kind of representation do you think students usually make to solve problems?" he answered, "Almost all students like to solve problems or justify their own claims using natural language because they have become accustomed to expressing their ideas in natural language in their daily lives." Although the math teacher encouraged students to solve problems using diverse representation modes, here, he emphasized natural language as a specific representation that was aligned with students' use of language in daily life. This may explain why none of his students used visual representation, although geometry is the most visual mathematical domain.

Some Typical Invalid Cases

It is a surprising finding that in this class, 15 students (about one half) presented failed argumentation, which can be divided into four categories: (1) no arguments given (four), (2) given question restated (five) (3) question misunderstood (six), and (4) mathematical misconceptions (two). For those students who offered no arguments or only restated the question, the math teacher (Mr. Chen) speculated that they had difficulty finding evidence to support a claim, which could be because they had little experience with this kind of question: "I think that students' thinking is rigid. They had few opportunities to access these kinds of questions in the examination, so they did not know how to deal with it when facing an unfamiliar question."

Students who misunderstood the question or existed a sort of carelessness tended to have difficulty with the statement, "if we cut the length to its half, the width is doubled, then the area of the rectangle will always remain the same." In the context of Chinese, the statement after "if" was a condition, and the statement after "then" was a conclusion. Therefore, both "cut the length to its half" and "the width is doubled" should be coordinative relation, but a few students thought they were the casual relation. For example, Song Qi (see Figure 5a) understood it as "after you cut the original length into half, then the width will look like it doubled, but it actually does not change," so he thought the area would become smaller. From his case, we can infer that Song Qi knew how to support his argument but misunderstood a critical statement or existed a sort of carelessness in the problem. These cases indicate the importance of the social perspective of conducting argumentation in the math classroom. With debating, students will gain opportunities to justify their understanding and reach a consensus before drawing on evidence supporting the claim.



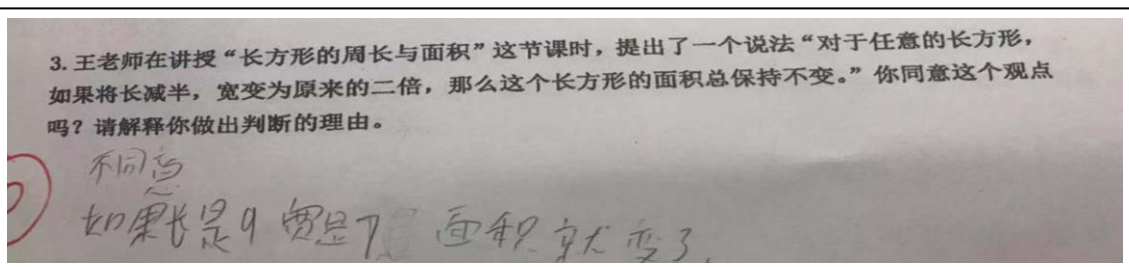
Song Qi's work (English version):

Don't agree.

Because the length was reduced to half and the width was kept the same, the width just looks like double-length as original, (therefore) the area of the rectangle must change, will decrease.

Figure 5a. Song Qi's Work Showing a Misunderstanding of the Question

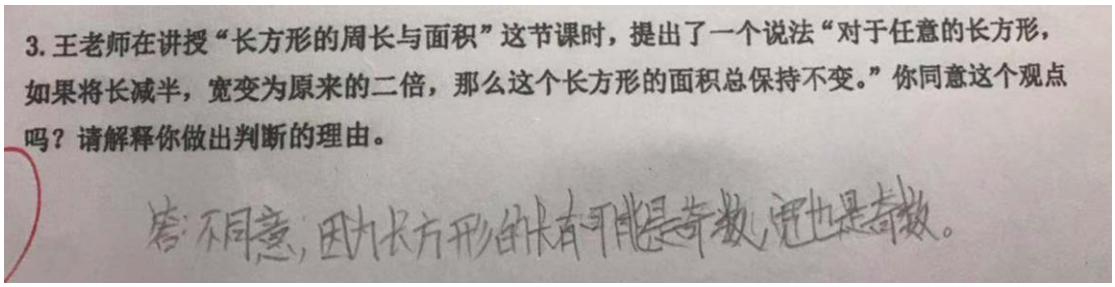
Students who based their argumentation on a misconception demonstrated students' difficulties in understanding some basic mathematics concepts. For example, Sun Xu's work (Figure 5b) and Zhao Fei's work (Figure 5c) their misconceptions about the concept of odd numbers. They might have thought that if the length is an odd number, after cutting it into half, it would be a decimal length, then the area must be changed. In the interview, the math teacher indicated his attention to students' need for problem-solving strategies, saying, "these students not only have some difficulty with this concept but also with other mathematics concepts, so giving them some useful ways of representation might be important in helping them deeply understand mathematics concepts."



Sun Xu's work (English version):

I don't agree with this view. If the length is 9 and the width is 7, the area will change.

Figure 5b. Sun Xu's Work Showing Mathematical Misconception



Zhao Fei's work (English version):

I don't agree. The length may be an odd number, and the width may also be an odd number.

Figure 5c. Zhao Fei's Work Showing Mathematical Misconception

Discussion

In this section, we discuss the results of our analysis of the performances of 33 students in one sixth-grade class on a geometrical argumentative task in combination with our interview with their math teacher.

Research Question Summary

In their responses to the problem, four presented strong arguments, about one third presented weak arguments, and nearly half of the students did not provide valid arguments. These results were similar to Vale, Widjaja, Herbert, Bragg & Loong's (2016) study of third and fourth-grade students' justifications, in which they found that most students couldn't give valid justifications, which they related to teachers' lack of noticing students' thinking and adequate teaching of argumentation. Students' differences in grade levels may partially explain the outcome differences. In the process of engaging in argumentation, students can experience the joy of sharing and develop their logic in mathematics (Stacey, 2010). However, our results differed from those of Lin and Tsai's (2012) study, in which most third graders could make justification by generalizable deductive arguments instead of empirical inductive arguments, perhaps because of differences in argumentative tasks. Whereas their argumentation task was to refute a false conclusion, our study was to prove a true conclusion. Also, a fifth-grade geometrical task may be more difficult than a third grade.

To further understand possible reasons for the students' performance related to an acceptable argument, we extended our research by interviewing their math teacher. The math teacher's response showed that argumentation is very important for students' mathematics development. But Mr. Chen may not prioritize creating an argumentative classroom culture for developing students' argumentation skills seems due to the limited teaching time. Mr. Chen's challenge in teaching argumentation, a kind of soft skill, for a mathematic-dense curriculum in China may also be a challenge for most math teachers in rural areas of China.

Among modes of arguments, the majority of the students used inductive rather, and some used deductive arguments and no students used a counterexample to solve the given problem. These results were similar to those of Widjaja, Vale, Herbert, Long & Bragg's (2020) study, in which 63% of the students used inductive

arguments, and only 7% used deductive arguments to solve a problem. They explained that the structure of the lesson allocated only enough time to conduct inductive arguments and not deductive arguments. However, the results in this study were not consistent with Lin and Tsai's (2016) study, whose explanation was the same as Flores' (2002) and Keith's (2006), which was that if teachers supported students in engaging in conjecturing, their justifications would be improved. In addition to these reasons, our interview showed that the teacher's argumentation knowledge seemed to be inadequate. His limited knowledge of deductive arguments and the assumption that students' cognitive development was not sufficient for them to conduct deductive argumentation may lead to his neglect of teaching deductive argumentation. Hence, only a few students used deductive arguments in solving the task. However, in some international studies, researchers have found that students' deductive argumentation can and should be developed in primary school (Lin, 2018; Stylianides, 2016; Van Ness & Maher, 2019), and they should be allowed to use informal mathematical language to express their ideas (Yore, Pimm, & Tuan, 2007). Thus, the interview further enriched the current research by demonstrating that teacher's own subject knowledge may also be a highly crucial factor in addition to lesson design and teacher support.

Among modes of argument representation, more than half of the students used natural language, and a third used both natural language and symbolic language to present their solutions. These results were consistent with Lin and Tsai's (2012) study showing that most students used the textual description or textual description combined with symbolic of displaying their solutions, but the researchers didn't give reasons for this phenomenon. Widjaja et al. (2020) also focused on the importance of representation in making argumentation but only distinguished it among oral, non-verbal, and written representations. However, we further explored diverse written arguments and reasons why students preferred textual description or textual description combined with the symbolic description in the interview with the math teacher, who thought that because students were accustomed to expressing their ideas in natural language in their daily lives, it was difficult for them to change. Although he realized the importance of representation and had encouraged his students to use different representations to solve problems, his students' performance had remained the same, suggesting that the math teacher may still need to find more effective methods to develop their students' representation abilities to support students in a diverse cognitive level.

In this study, some performance was aligned with the basic requirement of national mathematics curriculum standards. For example, "[students could] make observations, conjectures, experiments, and justifications in the practice of mathematics by using plausible reasoning [or logical arguments]" (Ministry of Education of the People's Republic of China, 2011). However, there were still nearly half of the students failing to demonstrate valid mathematical argumentation in the given task. These cases fall into four categories: no argument is given, the restatement of the given question, misunderstanding of the question, and mathematical misconception. With regard to understanding the meaning of the question, studies have shown that analyzing is the basis of generalizing and justifying (Reid, 2002; Jeannotte & Kieran, 2017). Thus, in argumentation teaching, the math teacher should develop students' ability to analyze questions.

In addition, some students misunderstood mathematical concepts such as the concept of odd or even.

Understanding mathematics concepts at each level is the basis for learning the next level of math (Clements & Sarama, 2004). Therefore, we further investigated this reason for students' failure to present valid argumentation in the interview with the math teacher, who thought students had lacked opportunities to engage in argumentation, so they were confused when faced with an unfamiliar argumentative task. Thus, more opportunities to engage in solving argumentation tasks are needed for students' development in argumentation (Stein, 2001).

Implications for the Math Teacher

First, from the perspective of teacher knowledge, we felt that Mr. Chen, a very experienced math teacher, should keep improving his subject knowledge to teach for the new curriculum standard. For example, he possessed limited knowledge of the modes of argumentation, especially deductive argumentation. The deductive argument, usually defined as a formal way of reasoning from the general to the particular (Harel, 2014), is an important method of argumentation even in primary school (Stylianides, 2016). Normally, in the elementary stage, students are not required to use formal mathematical language to express their deductive argument thinking. They can use informal mathematical language or natural language. Our examination of the six-graders' argumentation performance indicates that some of six-graders can conduct deductive argument, even under the context that their teacher, Mr. Chen, did not agree that deduction is important and have taught it purposefully. As such, if China wants to promote argumentation education in elementary school, which was suggested in the standard, offering some professional development plan regarding argumentation may be a good step, especially for rural experienced math teachers, who have implemented old math standards for decades.

Having a comprehensive knowledge of the nature of argumentation is the basis for teaching it (Simon, Erduran, & Osborn, 2006). In addition to the need to upgrade his content knowledge, the math teacher could improve the efficiency of his classroom teaching. He could design diverse argumentation tasks based on mathematical concepts and provide adequate help for his students at lower levels of performance in analyzing the meanings of questions and solving a question in various ways. In addition, the math teacher may also develop different argumentative tasks that provide a variety of opportunities for students to practice argumentation, both collaboratively and individually (Krummheuer, 2007).

Limitations and Future Research

This pilot study had some limitations with regard to the sample, which might be noted to inform future research. The sample in this study, comprising one class of sixth graders (33 students) in a rural school, cannot be considered representative of all Chinese sixth graders (Zhou, 2017). This is a pilot study to understand the argumentation performance of rural students, so future research should enlarge samples for a more comprehensive understanding of general trends of Chinese rural students' performance. Another limitation of this study was the use of only one teacher interview to pursue the reasons for students' performance, which provided a subjective perspective without other sources for verification. For future studies, nevertheless, we recommend multiple ways of determining underlying reasons for performance for cross-validation such as

conducting a survey or classroom observations.

Conclusion

This study showed that nearly half of the students in this rural sixth-grade classroom couldn't produce valid argumentation in solving one geometrical argumentative task. Of those who did, most used only one case (inductive argument) as evidence to justify their conclusion. Only a few students used general methods (deductive arguments) to justify their conclusion. Almost all students used informal language (textual description or textual description with mathematical symbols) to explain their arguments. Thus, this study reveals the performance of argumentation among students in the rural classroom needs to be improved. This finding is a matter of concern, as argumentation is fundamental to mathematical learning.

From the math teacher interview, we could identify ways in which the teacher's knowledge and practices might influence students' performance. First, the students lacked opportunities to construct arguments or solve argumentative tasks. Second, the math teacher's argumentation knowledge was inadequate enough to meet the requirements of the national curriculum standard. Third, he may need to seek more effective ways to develop students' ability to use diverse representations. In conclusion, teaching is a process of constant practice and exploration. For teachers like Mr. Chen, they may need to keep developing their knowledge about mathematical argumentation and constantly look for more effective strategies and methods for improving students' comprehensive argumentation abilities.

References

Balacheff, N. (1988). A study of students' proving processes at the junior high school level, *Paper presented at the 66th Annual Meeting of the National Council of Teachers of Mathematics*, U.S.A.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special. *Journal of Teacher Education, 59*, 389–407.

Chang, G. L. (2005). New mathematics curriculum reformation brings the realistic challenges which mathematics teachers have to confront. *Journal of Mathematics Education, 14*(2), 41-43.

Conner, A. M., Singletary, L., Smith, R., Wagner, P., & Francisco, R. (2014). Teacher support for collective argumentation: a framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics, 86*(3), 401-429.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*, 81-89.

Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers.

Corleis, A., Schwarz, B., Kaiser, G., & Leung, I. K. C. (2008). Content and pedagogical content knowledge in argumentation and proof of future teachers: a comparative case study in Germany and Hong Kong. *Z.D.M., 40*(5), 813-832.

Department for Education. (2013). *Mathematics: Programmes of study: Key Stages 1–2 (National Curriculum in England).* Retrieved October 31, 2013, from https://www.gov.uk/government/uploads/system/uploads/attachment_data/file/239129/PRIMARY_national_curriculum_-_Mathematics.pdf.

Dewey, J. (1902). *The child and the curriculum*. Chicago: University of Chicago Press.

Ellis, A., Özgür, Z., & Reiten, S. (2018). Teacher moves for supporting student reasoning. *Mathematics Education Research Journal, 31*(2), 107-132.

Flores, A. (2002). How do children know that what they learn in mathematics is true? *Teaching Children Mathematics, 8(5)*, 269-274.

Hanna, G., & De Villiers, M. (Eds.). (2012). *Proof and proving in mathematics education. The 19th ICMI study*. Dordrecht: Springer.

Harel G. (2014) Deductive Reasoning in Mathematics Education. In Lerman S. (Eds.), *Encyclopedia of Mathematics Education*. Springer, Dordrecht.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester, Jr. (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Greenwich, CT: Information Age Publishing.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31*(4), 396-428.

Hu, D. S., Wang, J., & Xu, H. W. (2013). Investigation of the Change of Mathematics Learning Way in the New Curriculum Reform Background. *Journal of Mathematics Education, 22*(5), 47-51.

Hunter, R. (2007). Can you convince me: learning to use mathematical argumentation. In J. H. Woo, H. C., Lew, K. S., Park, & D. Y. Seo (Eds.), *Proceedings of the 31st conference of the International Group for the Psychology of Mathematics Education,* Volume 3 (pp. 81–88), Seoul: P.M.E.

Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics, 96*(1), 1–16.

Kanellos, I., Nardi, E., & Biza, I. (2013). The interplay between fluency and appreciation in secondary students' first encounter with proof. In A.M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education (P.M.E.)* (Vol. 5, pp. 84). Kiel, Germany: P.M.E.

Keith, A. (2006). Mathematical argument in a second grand class: generating and justifying generalized statements about odd and even numbers. In S. Z. Smith & M. E. Smith (Eds.), *Teachers engaged in research: Inquiry into mathematics classrooms, grades preK-2* (pp. 35-68).

Krummheuer, G. (2000). Mathematics learning in narrative classroom cultures: Studies of argumentation in primary mathematics education. *For the Learning of Mathematics, 20*(1), 22-32.

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom. Two episodes and related theoretical abductions. *Journal of Mathematics Behavior, 26*(1), 60–82.

Kosko, K. W., Rougee, A., & Herbst, P. (2014). What actions do teachers envision when asked to facilitate mathematical argumentation in the classroom? *Mathematics Education Research Journal, 26*(3), 459-476.

Lin, P. J. (2018). The development of students' mathematical argumentation in a primary classroom. *Educação*

& Realidade (Education &Reality), 43(3), 1171-1192.

Lin, P. J., & Tsai, W. H. (2012). Fifth graders' mathematics proofs in classroom contexts. *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (vol.3, pp. 139-146). July 18-22. Wesley Girls High School, Taipei, Taiwan.

Lin, P. J., & Tsai, W. H. (2016). Enhancing students' mathematical conjecturing and justification in third-grade classrooms: The sum of even/odd numbers. *Journal of Mathematics Education, 9*(1), 1–15.

Liu, J. (2013). Curriculum reform in the new century: review and reflection. *Peking University Education Review, 11*(4), 1-19.

Livy, S., Herbert, S., & Vale, C. (2019). Developing primary pre-service teachers' mathematical content knowledge: opportunities and influences. *Mathematics Education Research Journal, 31*(3), 279-299.

Lu, S. H., Jiang, Y., & Li, Q. (2011). Survey on the new mathematics curriculum implementation in compulsory education. *Journal of Mathematics Education, 20*(5), 32-36.

Ma, Y. Q., & Cao, Y. M. (2017). International comparative study of geometrical reasoning in mathematical textbooks of six versions in middle school level. *Journal of Schooling Study, 15*(3), 52-65.

Martin, W. G., & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education, 20*(1), 41-51.

Martino, A. M., & Maher, C. A. (1999). Teacher questioning to promote justification and generalization in mathematics: what research practice has taught us. *Journal of Mathematical Behavior, 18*(1), 53–78.

Ministry of Education of the People's Republic of China. (2011). *Full-time Compulsory Education Mathematics Curriculum Standards.* Beijing: Beijing Normal University Press.

Nickel, O. (2019). Aspects of freedom in mathematical proof. *Z.D.M., 51*(5), 845-856.

Osborne, J. F., Henderson, J. B., Macpherson, A., Szu, E., Wild, A., & Yao, S. Y. (2016). The development and validation of a learning progression for argumentation in science. *Journal of Research in Science Teaching, 53*, 821–846.

Reid, D. (2002). Conjectures and refutations in grade 5 mathematics. *Journal for Research in Mathematics Education, 33*(1), 5–29.

Silverman, J., & Thompson, P. W. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education, 11*(6), 499-511.

Simon, S., Erduran, S., & Osborn, J. (2006). Learning to teach argumentation: Research and development in the science classroom. *International Journal of Science Education, 28*, 235–260.

Stacey, K. (2010). Mathematics teaching and learning to reach beyond the basics. In C. Glascodine & K. A. Hoad (Eds.), *Teaching mathematics? Make it count: What research tells us about effective mathematics teaching and learning* (pp. 17–20). Camberwell, V.I.C.: ACER.

Stein, M. K. (2001). Mathematical argumentation: putting umph into classroom discussions. *Mathematics Teaching in the Middle School, 7*(2), 110-112.

Stylianides, A. (2016). *Proving in the Elementary Mathematics Classroom*. Oxford University Press, Oxford.

Stylianides, A. J. & Ball, D. L. (2008). Understanding and describing mathematical knowledge for teaching: Knowledge about proof for engaging students in the activity of proving. *Journal of Mathematics Teacher Education, 11*, 307–332.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics*

Education, 38 (3), 289–321.

Stylianides, A. J. (2019). Secondary students' proof constructions in mathematics: The role of written versus oral mode of argument representation. *Review of Education, 7*(1), 156-182.

Vale, C., Widjaja, W., Herbert, S., Bragg, L. A., & Loong, E. Y. K. (2016). Mapping variation in children's mathematical reasoning: the case of 'what else belongs?'. *International Journal of Science and Mathematics Education,* 1-22.

Van Ness, C. K., & Maher, C. A. (2019). Analysis of the argumentation of nine-year-olds engaged in discourse about comparing fraction models. *Journal of Mathematical Behavior, 53,* 13-41.

Watson, A., & Ohtani, M. (Eds.). (2015). *Task design in mathematics education: An ICMI study 22*. Switzerland: Springer.

Widjaja, W., Vale, C., Herbert, S., Long, E. Y. K., & Bragg, L. A. (2020). Linking comparing and contrasting, generalizing and justifying: a case study of primary students' levels of justifying. *Mathematics Education Research Journal,* https://xs.scihub.ltd/https://doi.org/10.1007/s13394-019-00306-w.

Yore, L. D., Pimm, D., & Tuan, H. L. (2007). The literacy component of mathematical and scientific literacy. *International Journal of Science and Mathematics Education, 5*, 559–589.

Zhou, D. (2017). A survey of the fourth graders' mathematics performance: based on the east, middle and west regions of China. *Journal of Shanghai Educational Research, 4*, 47-51.

Author Information

Da Zhou

https://orcid.org/0000-0002-9463-6637

Northeast Normal University

Changchun

China

Contact e-mail: *zhoudadepict@163.com*

JinQing Liu

https://orcid.org/0000-0001-7818-5875

Indiana University

Bloomington

United States of America

Jian Liu

https://orcid.org/0000-0001-7501-6881

Beijing Normal University

Beijing

China