Geometric Refutations of Prospective Secondary Mathematics Teachers

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Article Info	Abstract
	Increasing students' exposure to mathematical reasoning (Lannin, Ellis, & Elliot, 2011) has been a constant call from mathematics education reform efforts. However, studies have raised concerns about whether teachers are prepared to teach reasoning. This paper explores one aspect of reasoning, refutations, by examining the geometric refutations of thirteen pre-service secondary teachers (PSTs). Findings suggest that the PSTs did struggle to create culturally accepted refutation, but their results were less concerning than previous studies suggest. Primary challenges include that the PSTs separate the work they do for the different subprocesses of reasoning, reason differently depending on the subprocess, and favor non-culturally accepted means of refutations. Additionally, when the PSTs created culturally accepted refutations, they often used specific counterexamples, which are less pedagogically powerful than general counterexamples (Zaslavsky & Peled, 1996). However, the PSTs in this study did provide general counterexamples more often than the participants of other studies (Potari, Zachariades, & Zaslavsky, 2009). Suggestions for ways to improve PSTs understanding of reasoning are provided.

Introduction

U.S. students have routinely struggled with tasks that require reasoning on national and international assessments (Arbaugh, Brown, Lynch, & McGraw 2004; Mullis, Martin, & Foy, 2005; Roach, Creager, & Eker, 2016). Due to these struggles, a cornerstone of mathematics education reform has been to call for an increase in mathematical reasoning (Fey & Graeber, 2003; National Council of Teachers of Mathematics [NCTM], 2000), where mathematical reasoning (hereafter reasoning) is often thought of as a non-linear, inter-related process of conjecturing or generalizing, investigating why, and validating or refuting (Lannin, Ellis, & Elliot, 2011). In the U.S., many states have adopted the Common Core State Standards ([CCSS] NGA/CCSSO, 2010) and the included *Standards of Mathematical Practice* that emphasize the role of reasoning in mathematics education. The shift in focus to include more reasoning is promising for many reasons. However, research has shown that many in-service and pre-service teachers struggle with reasoning themselves (Knuth, 2002). Thus, teachers may not have sufficient knowledge of reasoning to teach it at the level suggested by the CCSS. To help prepare future teachers, we, as teacher educators, need to provide future teachers with opportunities that not only support their learning of reasoning, but that also support their ability to create productive environments that help their

students learn to reason. To do that effectively, requires understanding the ways that teachers currently reason so that we might build up toward more sophisticated forms of reasoning (Heid, Wilson, & Blume, 2015).

The goal of this paper is to examine one subprocess of reasoning, refutations, and in a particular domain of mathematics, geometry. Specifically, this study seeks to answer the question, how do secondary pre-service teachers refute geometric conjectures. The goal in answering this question is to provide suggestions for how to better prepare future teachers to take on the challenges that an increased focus on reasoning will require.

The Role of Refutations in Mathematical Discovery and Epistemology

Refutations are the focus of this paper because they should be common in mathematics. The justification for this is three-fold. First, refutations play a critical role in mathematical discovery and epistemology. Additionally, studies have shown that elementary-age children can engage in refutations (c.f., Balacheff, 1991; Komatsu, 2010; Lampert, 1990; Larsen & Zandieh, 2008; Reid, 2002) and that doing so resulted in learning (c.f., Komatsu, 2010; Larsen & Zandieh, 2008). Each of these claims is explored in more detail next.

Refutations often, but not always, take the form of a counterexample, and Lakatos (1976) described three ways that mathematicians address counterexamples. The first is monster-barring, where one treats the counterexample as an exception to the meaning of the concept the counterexample refuted. The next was exception-baring, where the counterexample was treated as an exception to the theorem. The final method was what he called proofs and refutation, which followed the pattern of conjecture, proof, counterexamples, and proof re-examined (Lakatos, 1976). It might be common to think of the proof as the culminating, and therefore most important part of the reasoning process, but Lakatos demonstrated that refutations often served as the driving force of mathematical discovery by serving three important purposes. Refutations refine conjectures because they force the reasoner to define the largest set of objects for which the conjecture is true. They expand the meaning of concepts by creating more distinct boundaries between what the reasonser includes as an example and non-example of the concept. Lastly, refutations can improve proofs by making unjustified claims explicit. In essence, Lakatos demonstrated that refutations were important for proof and concept development, two critical features of quality mathematics instruction.

Refutations have also been considered an important part of epistemology. Balacheff (1991), for example, noted the similarities between Lakatos's work and Piagetian constructivism. Particularly, a contradiction causes a state of disequilibrium in an individual and that by overcoming said contradiction the individual learns (Piaget, 1971; von Glasersfeld, 1995). To make students aware of contradictions and in turn lead to a conceptual advance, Balacheff (1991) noted, "the teacher's interventions will be fundamental. The way he or she manages the teaching situation may bring the students to see that their knowledge and the rationality of their conjectures must be questioned and perhaps modified" (p. 109). From a different epistemological perspective, Boero, Garuti, Lemut, and Mariotti, (1996) coined the phrase "cognitive unity" to describe the continuity that exists between the creation of a conjecture–often created through examination of examples and counterexamples–and the potential for a proof of that conjecture to be constructed. Boero and colleagues (1996) argued that separating

the processes of conjecturing and proving breaks that unity and hinders proof development. They suggest that to develop students' understanding of reasoning, instruction ought to take advantage of that unity and thus build from students' conjecturing. Again, conjectures are often created by examining examples and counterexamples.

Despite the previous claims, it seems valid to question whether students are even capable of creating refutations. Studies have shown that students in elementary and middle grades are capable of following similar patterns of reasoning as those described by Lakatos (Balacheff, 1991; Komatsu, 2010; Lampert, 1990; Larsen & Zandieh, 2008; Reid, 2002). These studies lack in their ability to generalize to traditional classroom practices as they were conducted in non-typical settings (e.g., two students per teacher or the researcher served as the instructor). However, their work does at least serve as an existence proof that even very young children are capable of this type of reasoning (Balacheff, 1991; Komatsu, 2010; Lampert, 1990; Reid, 2002). Additionally, those refutations did play an important part in the students' learning (Komatsu, 2010; Larsen & Zandieh, 2008). In summary, students, even very young students can create refutations, and there are historical and epistemological justifications that refutations lead to learning.

Theoretical Framework

Refutations frame the work in this study, and the meaning used in this study is connected to the work of proof schemes and proof productions. In this paper, refutations take on a psychological and not mathematical interpretation. The definition of refutation and justification for this choice will be elaborated in this section.

Refutations Connection to Proof Schemes and Proof Productions

It might seem odd that a paper on refutations is informed by proof, but instead of seeing them as opposites of the same coin, I see them, as Lakatos (1976) did, as influencing each other. In this paper, a refutation is meant to be any argument that convinces an individual or community that a conjecture is false. This definition should be interpreted in the same psychological sense as Harel and Sowder's (1998) proof scheme. Meaning that the types of arguments the participants used to convince themselves that conjectures were false may not necessarily align with the types of arguments that members of the mathematics community would accept as refuting a claim. This psychological interpretation was chosen to better understand the reasons the PSTs used arguments that are not considered refutations by members of the mathematics community. Additionally, this decision provides a description of what the PSTs can do versus what they cannot do in terms of creating a refutation.

The PSTs in this study were asked to explain why they felt the conjectures were false, thus producing an argument was an expectation. Weber and Alcock (2004) suggest there are two ways of constructing a formal proof, which they called syntactic and semantic proof productions. A syntactic proof production is "one which is written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way" (p. 210). Whereas a semantic proof production "uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210). An instantiation is "a systematically repeatable way that an individual thinks about a mathematical object, which is

internally meaningful to that individual" (p. 210). A key difference between syntactic and semantic reasoning is that semantic reasoning focuses on an instantiation, which need not be but often is an example. However, this is not to claim that semantic reasoning is inductive as one might examine examples critically looking for patterns of variation and make conjectures or generalizations based on those insights. This makes semantic reasoning akin to Harel and Sowder's (1998) description of the transformational proof scheme which they described as a foundational aspect of deductive reasoning. Syntactic and semantic seem to be contrasting ways of reasoning, but as we will explore in this paper, it may be beneficial to use both.

To describe refutations in their entirety is beyond the scope of any one paper as the type of refutation used depends on the type of conjecture being refuted (e.g., universal or existential conjecture). The conjectures explored in this study were exclusively universal conjectures (i.e., for all X, there is a property Y) and conjectures of this type are commonly refuted using a counterexample. In fact, counterexamples are so common that even Lakatos (1976) only distinguished between local counterexamples (refuting specific lemmas used in the proof) and global counterexamples (refuting the main conjecture). However, Peled and Zaslavsky (1997) described two important types of counterexamples, specific and general. Their reason for the distinction was pedagogical. Specific counterexamples simply refute conjectures, whereas general counterexamples explain why the conjecture is false. General counterexamples are more powerful pedagogically because they provide insight into why the conjecture is false. This insight then provides further information about how to amend the conjecture and continue the reasoning process.

Counterexamples, however, are not the only means for refutation. An argument can be used to describe how some or all the conditions of the conjecture lead to a logical absurdity, making the conjecture false. This is part of the mathematical practice known as reductio ad absurdum (RAA). Weber (2009) described an unusual case of a student, Isaac, who refuted conjectures by reasoning in a purely syntactical way instead of the traditional semantic way. That is, Isaac used arguments instead of counterexamples. Isaac would convert the statement into a statement of first order logic, try to prove the conjecture, and only after seeing why he reached an impasse would he suggest a counterexample. This work is relevant for two reasons. First, it offers another way that conjectures can be refuted other than simply examining examples, semantic reasoning. Secondly, Weber (2009) noted that some advanced, college-level students and mathematicians predominately use syntactic reasoning, which lead him to wonder whether training in semantic reasoning would lead Isaac to more robust ways of reasoning or simply confuse him. Additionally, Weber (2009) wondered whether more visual concepts, like geometry covered in this study, would provide insights into this question. These questions are explored in the discussion.

Classifying Refutations

Previous studies have examined how individuals create refutations, and these studies can be put into two groups which varied by the participants. This section summarizes the findings from these studies to create a taxonomy of refutations. Additionally, the work of Sekiguchi (1991), who examined how one high school geometry teacher engaged her students in proof analysis, is briefly summarized.

How Secondary Students Refute Conjectures

Three studies examined how school-age students refute conjectures. Hoyles and Küchemann (2002) surveyed over 2500 English students in Year 8 (age 13) and the same students again in Year 9 from high attaining mathematics classes of randomly selected schools. Lin (2005) surveyed 1,181 seventh and 1,105 eighth-grade Taiwanese students. While Lee surveyed 60 students ages 14-15 in Singapore on two separate occasions.

Hoyles and Küchemann (2002) presented the participants with an algebraic conjecture that they claimed was not familiar to the participants. They asked them to decide if the conjecture was true or false and asked them to explain why. They found that 36% of Year 8 students were able to give a correct counterexample where 28% of the counterexamples were specific and the rest were general counterexamples. Of the incorrect responses, 18% assumed the statement was true and confirmed it with examples. They did note "discernable but modest progress from Year 8 to Year 9" (p. 204). However, they only included the categories of specific counterexample, general counterexample, and miscellaneous incorrect.

Using adapted versions of Hoyles and Küchemann's (2002) items along with some additional items, Lin (2005) found that there was no evidence of improvement in the participants' assessment of a conjecture's validity from seventh to eighth grade and students from both grades did roughly as well as the students in Hoyles and Küchemann's (2002) study. Lin (2005) coded the work of the students who attempted to refute the conjectures into three categories: rhetorical argument, correcting the given information, and generating counterexamples. The use of rhetorical arguments was unclear. The only description given was that the reasons are relative to the person creating the refutation. Generating counterexamples included both specific and general counterexamples as well as non-examples. Very few students created general counterexamples (less than 4%) and fewer yet gave a proof (less than 1%). Interestingly, the most common method of refutation was what Lin called providing an alternative Q (for a conjecture in P $\rightarrow$ Q form where P is the hypothesis and Q is the conclusion). This was a sub argument of correcting the given information. Essentially, the nature of this type of refutation was to suggest because a different but related conjecture is true, the current conjecture is not. Hereafter this type of refutation is referred to as providing an alternative conjecture.

Lee (2016) gave his participants two false conjectures, and asked them to decide if the conjectures, written in "if-then form", were true or false and to justify their conclusions. For false statements, they were asked to find an object that satisfied the conditions but failed to meet the conclusions–create a counterexample. Lee categorized the participants refutations into six, hierarchical levels, where the top two levels were considered culturally accepted means for refutations–specific and general counterexamples. The non-culturally accepted means of refutation were similar to those described earlier. He found that roughly 50% of the students were able to create mathematical refutations for the conjectures.

It is clear from these studies that students from a variety of ages and cultures struggle to create refutations. Many of these students struggles may be a result of the instruction they receive. Although Sekiguchi's (1991) study cannot generalize to all cases, in many ways his study could be considered as a best-case scenario as the

teacher was experienced, knowledgeable, and was teaching an advanced class of students. When refuting students' statements, the teacher rarely used counterexamples (the use of counterexamples was observed only twice). Thankfully, Sekiguchi (1991) did notice that students refuted statements, but again he called it rare. This is despite the fact that indirect proof was a section studied in the class during the study. The teacher's refutation relied on authority, the acknowledgement of the violation of a classroom norm, or that a theorem, definition, or formula used did not apply to the current situation. Providing an alternative conjecture was a common form of refutation for the teacher despite it not being a mathematically acceptable refutation. Interestingly, Sekiguchi (1991) commonly observed both the teacher and the students engaging in normative forms of proof, but rarely normative forms of refutation. Knuth (2002) similarly suggested that because the teachers in his study over emphasized the role of verification for proof, they might overly focus on teaching proof as a means for verification. Thus, students' struggles with proof may be a result of the fact that refutations are often given less treatment in the curriculum, and when refutations are covered, they are not used in normative ways. Studies of how pre-service and in-service teachers refute conjectures suggest that teachers' understanding of refutations may be a cause of this. These studies are explored next.

How Teachers Refute Conjectures

Three studies have looked at how in-service and pre-service teachers refute conjectures. Potari, Zachariades, and Zaslavsky (2009) examined how 76 candidates in a master's in mathematics education program (6 elementary, 70 secondary, and 36 in-service) refuted conjectures. The participants were asked to respond in writing to five hypothetical teaching events, one of which involved a refutation. Given a specific geometric task, two hypothetical students developed conjectures, and the participants were asked how they would respond if the dialogue took place in their classrooms. Of the 76 teachers, three did not reply to the task, eight considered the false conjecture true, and two others gave an implicit justification for the conjecture. Of the 63 remaining teachers, 36 (47%) of their refutations were not culturally accepted refutations. Twenty teachers created incorrect counterexamples and 16 created "proof-like" arguments to refute the conjecture. In these arguments, the teachers claimed that none of the theorems about triangle congruence applied to the case the teachers were presented, thus making it false. Potari and colleagues (2009) referred to this type of reasoning as over-reliance on familiar criteria, a term I adopt. Refutations of this type essentially state that because no known theorem proves this conjecture it is false. Unfortunately, it is common for these teachers to create invalid refutations, and uncommon for them to create a valid refutation. Thirteen teachers (17%) created valid refutations (2 proofs, 11 counterexamples).

Giannakoulias, Mastorides, Potari, & Zachariades (2010) gave 18 secondary school mathematics teachers a questionnaire with three calculus problems with responses by hypothetical university students that contained errors. The PSTs were asked to identify the errors and provide an explanation to help the student understand their error. They found "theory" was the most common refutation, whereby theory meant three different types of refutations. The first, was similar to the reasoning described by Potari et al. (2009) where the PSTs claimed there was no such theorem to prove it. The second type of theory refutation was to claim the theorem was applied inappropriately, and the third type was to give a general rule (e.g., you cannot divide by zero).

Giannakoulias, et al. (2010) claimed these types of refutations are problematic pedagogically because it fosters the authoritarian proof scheme (Harel & Sowder, 1998) where students develop a belief that they should accept the teacher's statements without questioning them. They interviewed a few participants to further investigate this finding and they found that the PSTs preferred these types of refutations to counterexamples because they considered the counterexample to be a specific case whereas the "theory" refutation provided conditions under which the claim can be valid. Although this could be considered a misunderstanding on the teachers' part, justifications for thinking this way are explored in the results section.

Finally, Zaslavsky and Peled (1996) gave 36 experienced in-service secondary teachers and 67 pre-service secondary teachers false statements supposedly created by a student and were told that they were false. They were asked to create "at least one example to convince this (supposed) student that the statement was false [and] … to specify in detail the process that led them to their examples" (p. 70). Only 33% of the in-service teachers and 4% of the preservice teachers were able to create at least one complete, well-justified, and correct counterexample. However, the in-service teachers did outperform the preservice teachers in many ways leading Zaslavsky and Peled (1996) to suggest that the in-service teachers had developed this knowledge out of necessity while teaching.

Three findings from these studies stand out as relevant for the current study. First, many individuals (school-age children and their teachers) have difficulty determining the validity of mathematical conjectures. Second, if they correctly determine a statement to be false, it is rare for them to construct a valid mathematical refutation. Lastly, when a mathematical refutation is created, it is frequently a specific counterexample. The codes from the studies described above informed the coding process for this study described in the next section.

Method

The data presented here are taken from a larger project (Creager, 2016) involving thirteen pre-service secondary teachers (PSTs). All PSTs were near the end of an undergraduate program for secondary mathematics education at a large Midwestern university. The data was collected during the fall semester. There were three areas of data collection: a survey, and two interviews. For the sake of space, I briefly describe the survey and the nature of the second interview and follow this with a more elaborate description of the first interview which is the focus of this paper.

The survey consisted of 13 multi-part items that covered geometric knowledge (e.g., evaluating definitions and conjectures and determining properties of triangles and quadrilaterals), geometric reasoning (i.e., they were asked to evaluate whether various arguments were proofs), and knowledge of logic in a geometric setting. For the latter, students were given a scenario where one student claimed a conjecture was true, while another claimed the converse was true. The PSTs were asked to determine if one both or none of the fictional students' statements were correct and to justify why they thought that. Both conjectures were written in the traditional "if-then" form. The participants were asked to complete this before participating in the interviews. The first interview was about one month after they submitted the survey to allow for a preliminary analysis of their data.

In the second interview, the PSTs were presented with a teaching scenario. This interview took place about one month after the first interview to allow time for a preliminary analysis of the first interview. The participants' goal for the interview was to plan a lesson that focused on the teaching of proof for a particular high school geometry content standard. The PSTs were given several researcher-generated arguments that ranged from empirical to deductive. Participants were told that the arguments were found online. The PSTs were then asked to design a lesson and could use any combination of the arguments in their lesson plan or create their own. Although the focus of this paper is on the PSTs' responses in the first interview, the data from these sources informed the analysis on the PSTs' proof schemes and geometric knowledge (see Creager, 2016 for more description). This interview lasted roughly one hour.

The first interview was an hour-long, cognitive interview where the PSTs were asked to evaluate, create, and prove conjectures. To initiate this process, they were given a list of thirteen geometric conjectures (see Table 1).

Table 1. Conjectures Used During the Geometric Proof Interview

#	Conjecture	Validity
1	The diagonals of a quadrilateral cut the area in half.	False
2	The diagonals of a rectangle bisect the vertex angles.	False
3	The diagonals of a rhombus are angle bisectors.	True
4	The midsegment of a quadrilateral is parallel to the remaining sides.	False
5	The median of a triangle creates two congruent triangles.	False
6	An angle bisector of a triangle is perpendicular to the opposite side.	False
7	The diagonals of a rectangle are congruent.	True
8	A line that bisects an angle of a triangle also bisects the side that is opposite the angle.	False
9	Isosceles triangles are congruent when their vertex angles are congruent.	False
10	When any point on an angle bisector of a triangle is connected to the opposite vertices of the triangle the resulting triangle is isosceles.	False
11	A median of a triangle divides the triangle into two smaller triangles of equal area.	True
12	The diagonals of a parallelogram bisect each other.	True
13	The diagonals of a trapezoid are congruent.	False

Four of the conjectures were true, while nine were false. The PSTs were told that the conjectures are based in Euclidean or planar geometry and that some are true, and some are false. They could pick which conjectures they worked on and told to pick those with which they felt comfortable. Once they had picked a conjecture, they were asked the same series of questions. First, they needed to decide if the conjecture was true or false. If they felt the conjecture was true, they were asked to prove it. If they felt the conjecture was false, they were asked to show why it was false and amend the conjecture so that it would be true.

Finally, they were then asked to prove their amended conjecture. The PSTs occasionally amended the conjecture in a way that was again false, and then later realized this. Thus, refutations for conjectures they created were also possible, and these refutations were considered during the analysis. The conjectures the PSTs were given

were not written in the typical conditional form of "if-then" statements because a secondary goal of the research study was to learn about how the PSTs interpreted statements of inference in everyday language. The implications of how they interpreted the statements are also discussed. Conjectures of this type have been used in many previous research projects, the oldest of which to my knowledge was Dreyfus and Hadas (1987) and their reason for this structure mirrors my own. Namely, the conjectures are not obviously true or false as individuals often see little need to prove or refute statements that are obviously true or false. Additionally, this problem structure required that students go through the full reasoning process (Lannin et al. 2011) versus doing only one aspect of it (e.g., prove this statement or find a counterexample for this statement, etc.).

The purpose of the first interview was to learn about the PSTs' geometric reasoning abilities. To ensure that the time was well spent, the interview included three additional features. The first was that they could pick which conjectures they worked on and were encouraged to choose those with which they felt comfortable.

Table 2 describes which conjectures were worked on by the participants. On average the PSTs spent about six minutes per conjecture. The second feature was that they were provided a definition sheet that included definitions for the terms used in the conjectures that were taken from the glossary of a high school geometry textbook (Charles, et al., 2011). Finally, the participants had at their disposal a ruler, compass, protractor, and computer with Geometer's SketchPad (GSP). The version of GSP they were given had a tools file uploaded that could construct the different classes of triangles and quadrilaterals. All the participants had experience with GSP from a previous course, and they were shown how to use the tools file.

Table 2. The Conjectures Worked on by Each Pre-Service Teacher

	Conjectures												
Name*	1	2	3	4	5	6	7	8	9	10	11	12	13
Alana	x		x	x	x	x			x				x
Carol	x	x	x		x	x	x		x				
Elisa	x		x	x	x		x						
Haley	x		x		x		x		x		x		x
Lacy					x		x		x		x		
Lucinda	x				x	x	x		x			x	
Kade	x	x	x		x		x					x	
Karla	x	x	x			x	x						x
Kay	x		x		x	x			x		x		
Macy	x	x		x									
Mike	x	x					x		x				
Sabrina	x		x	x						x	x		
Sadie	x		x				x		x				x

* All names are pseudonyms

Participants and Data

The participants included two males and eleven females. One PST was a member of a program that included an extended field experience of four semesters with one mentor teacher. Another PST was majoring in special education and getting an endorsement in mathematics. The remaining eleven participants were enrolled in a traditional teacher education program. All the PSTs were enrolled in the second of two methods classes and were all in their third or fourth year of a secondary education program at a large, research-oriented, public university in the Midwest. These participants represented a purposeful sample (Creswell, 2009) because they had either finished or were nearly finished with their mathematics requirements, meaning that formal training in mathematics prior to teaching was complete or nearly so.

The data for the study come from audio and visual recordings of the interviews. There was one camera focused on the PSTs' work and one camera focused on the interaction between the interviewer and the participant. Transcripts with thick descriptions (Carspecken, 1996) were made from the recordings that included gestures and expressions. Additionally, any work they performed during the interview was scanned and included in the transcripts.

Data Analysis

The data was first broken done for each participant by conjecture and by the type of subprocess of reasoning they were working on (i.e., conjecturing/generalizing, investigating why, or proving/refuting). Although the line between this work can be blurry in many settings, it was often quite clear in the interview because of the structure of the interview. The participants were asked to think aloud, and so students would say whether they were investigating the conjecture or trying to prove/refute them. Additionally, I often asked them specifically if they were making sense of the conjecture or trying to prove/refute it. Lastly, when students finished their work on a conjecture, I asked them to explain their process and it was common for them to say that their work at one point was meant to explore the conjecture whereas at other times they were proving/refuting it. Because the participants were also asked to provide an alternative conjecture for the conjectures they felt were false, it was not uncommon for the PSTs to create a new, but still false conjecture. Occasionally the PSTs realized this and refuted this conjecture. The conjectures they created were considered new conjectures and their work was broken up as described above. Finally, it was not uncommon for a PST to set out to prove a conjecture, realize it was false and refute it or vice versa. These portions of the transcript were coded as both proving and refuting in these cases.

The focus of this paper is on the PSTs' refutations, which were analyzed using the codes from the studies summarized above. However, I remained opened to the option of new codes as well. Table 3 summarizes the codes from the studies described above that were found in the data for this study. One important distinction arose in categorizing the types of arguments the PSTs created to refute conjectures. The PSTs arguments often fell short of being reductio ad absurdum because they lacked in formality. However, the degree of formality that

was lacking ranged from minimal to quite significant. Thus, two codes, formal and informal arguments were created to capture this distinction, and both are described in the section that follows.

Table 3. Description of the Types of Refutations Found in Previous Research

Culturally Accepted Refutations
Specific Counterexamples – an example that refutes the conjecture
General Counterexamples – a counterexample that explains why the conjecture is false
Reductio Ad Absurdum – demonstrating that the conjecture leads to an absurdity, thus proving it is false
Non-Culturally Accepted Refutations
False Counterexamples – an example that does not refute the conjecture
Over Reliance on Familiar Criteria – suggesting a conjecture is false because a proof cannot be created using common theorems
Faulty Logic – an argument that does not refute a conjecture because it contains a logical flaw
Providing an Alternative Conjecture – suggesting that the conclusion of a conjecture is not true because an alternative, and assumingly contradictory, conclusion is true

Results

The refutations the PSTs created are broken into two sections. First, the overall results are presented and compared with those of other studies of PSTs' refutations. In the second section, their refutations are described in detail along with descriptions for why they created them. These descriptions are broken down by the result of that argument, whether it was semantic (i.e., based on counterexample) or syntactic (i.e., argument based).

Overall Results

The PSTs in total refuted 48 conjectures where some of the conjectures they refuted were conjectures that they had created. Only twice did a PST create multiple refutations for one conjecture, so both refutations were counted. Both times, neither refutation was a culturally accepted refutation. Table 4 includes the counts and percent of the refutations the PSTs created. The majority of the refutations the PSTs created (60%) included both a counterexample and an argument to support that counterexample. Although this is likely a result of the structure of the interview, it does show evidence that the PSTs are capable of this type of reasoning. Fifteen (31%) of their refutations included a counterexample only, and for all but six (13%) of those times it was a general counterexample. Thus, when the PSTs did not create an argument, they typically created general counterexamples which are pedagogically powerful counterexamples. Moreover, only three times (6%) was an argument given without a counterexample. In all three of those cases the refutation used the reasoning pattern over reliance on familiar criteria. Two additional positive notes include that only four conjectures (8%) were refuted with a false counterexample, and general counterexamples were common. This suggests that when given the opportunity to express their reasoning the PSTs did so, which differs from the findings of Potari and colleagues (2009). However, only nine (28%) of the arguments created by the PSTs were culturally accepted

ways of refuting conjectures, these are referred to as formal (7) and informal (2). Before providing examples of the types of refutations, how the PSTs interpreted the conjectures is explored.

Table 4. Types of Geometric Refutations Created by Secondary Pre-Service Teachers

Refutation Type			Number	Percent
Counterexamples			45	94
	Specific		19	40
	General		22	46
	False		4	8
Arguments			32	67
	Culturally Accepted		9	19
		Formal	7	15
		Informal	2	4
	Not-Culturally Accepted		23	48
		Providing an Alternative Conjecture	12	25
		Faulty Logic	6	13
		Over Reliance on Familiar Criteria	5	10

For the most part, the PSTs were clear in how to interpret the statements. Many of the PSTs wondered whether the conjectures were meant to be sometimes true, always true, or never true questions. After explaining that I was interested in whether they could prove the statements were true or false, they understood the statements to mean always true. Lucinda, however, held on to this confusion throughout the interview, but she also demonstrated a weak understanding of proof. It might have been that she understood the statements were meant to be always true but was unaware of how to answer them mathematically. The conjectures being in everyday language and not a traditional form, such as "if-then" form did not seem to hinder their thinking. Not once did a PST prove the converse or work in a way that would have suggested they misunderstood the logic of the statement. Many PSTs did struggle to answer in a purely mathematical way because the question was posed in a teaching scenario–the conjectures were suggested by a fictitious student. The PSTs wondered whether students would understand certain properties in their proofs or refutations, whether their language was age appropriate, and whether they were being overly critical. However, it was easy to steer them back to thinking in a purely mathematical way. Thus, their pedagogical considerations did not affect the analysis of their refutations, which are summarized next.

"That's it. Done!" – Counterexamples

Counterexamples were the most common form of refutation. On a positive note, the PSTs rarely created false counterexamples (4) and often created general counterexamples (22). However, specific counterexamples (19) were still common despite the fact that they were asked to explain how they knew their counterexample refuted the conjecture. Sadie only created one specific counterexample, but she most clearly articulated a possible reason for why the specific counterexamples were commonplace.

For conjecture 13, Sadie asked some questions to clarify the conjecture. After doing so, she drew a valid counterexample and said the conjecture was false. When I asked how she knew the conjecture was false, she said, "This diagonal is longer" and pointed to the longer diagonal. When asked again to explain why the conjecture was false, she said, "If I'm disproving something, and if I can find one example to disprove it, then it's clearly not true for everything. So, I feel that's it. Done!" Sadie's statement showed a clear break for her in the work she expected to do in the reasoning process once she reached a counterexample. Refutations were the end of her reasoning process. Other PSTs expressed that refutations were the end of their reasoning process in three other ways.

The first piece of evidence that counterexamples were the end of the PSTs' reasoning was that eight of the PSTs expressed feeling confused when asked to provide an explanation for their counterexamples. Lacy asked if I was asking her a trick question, while the others asked me to clarify what I meant by the question. The second piece of evidence that refutations were the end of their reasoning came when they amended the conjectures. The eight PSTs in this group often moved along classes of figures named in the conjecture. For example, while amending conjecture 1, Carol named several classes of quadrilaterals that made the conjecture true instead of examining key features that cause the conjecture to be true. This was not a result of the PSTs having non-hierarchical images of quadrilaterals because they correctly answered questions on the survey about the hierarchical nature of quadrilaterals. Instead, the PSTs seemed to be starting a new reasoning process where they were exploring and conjecturing. More specifically, they did not use the information they had gained while refuting to inform their conjecturing/exploring process. This resulted in difficulties with amending the conjectures for these eight PST. Their difficulties amending conjectures is described in more detail below in the section "That's Only True When …". The final evidence to support that the creation of a counterexample was the end of their reasoning process was that the arguments the PSTs created to refute the conjectures often used non-culturally accepted means. Thus, they seemed to be confused about the need to create an argument to refute a conjecture, and when they did it was often a weak point for them. Their proofs were more likely to be culturally accepted proofs than their refutations. This is contrasted by the work of Mike presented next.

Argument-Based Refutations

The participants typically (63%) provided an argument to support their counterexamples, and this was likely a result of their being asked to explain their counterexamples. However, it does suggest that the PSTs could reason this way when provided an opportunity. This is an improvement from the participants in Potari and colleague's (2009) study where participants offered an argument for only 29% of their refutations. Unfortunately, the arguments the PSTs created were not always sufficient to be considered a mathematical refutation. Examples of the arguments the PSTs created are given next.

"I'd Like To Prove It's False" – Mike's Use of Reductio Ad Absurdum

Mike regularly tried to create arguments to explain why the conjectures were not true. While working on conjecture 2, Mike used reductio ad absurdum (RAA) to refute the conjecture. He drew a picture and then

decided to use GSP to construct a rectangle. I asked how he was thinking about the problem, and he replied, "I'm just kind of looking at it and thinking of something to do. I can see that it is false." At this point, it was clear that Mike's counterexample perceptually refuted the conjecture for him. He continued, "I'd like to be able to prove that. I can see that it's false, but I can't state why it's false." Mike wanted to be able to prove that the conjecture was not true, which for him meant explaining why it is not true. Doing so however, was not easy for Mike. After several minutes, he admitted being stuck, and I suggested he could move on if he felt it that it was not a productive use of time. He next wrote a valid conjecture after which he had a stroke of insight, which is captured in Data Excerpt 1.

Data Excerpt 1. Mike's Reductio Ad Absurdum Argument for Conjecture 2.

Mike: Oh! I might know why. I can do this [refute conjecture 2] now.

Interviewer: Okay.

Mike: It involves corresponding angles. Since this [drew ABCD in Figure 1] is not a square this long side does not equal the short side. Thus, the corresponding angle to the long side is larger than the corresponding angle to the short side. Since we know that this angle DBC is greater than angle BDC and these two lines [passed pen over AD & BC] are parallel we know by some theorem I can't remember that this angle [ADB] is equal to that one [DBC].

Int.: Okay, hang on, I got lost. Go again.

Mike: Since this side [pointed to DC] is longer its corresponding angle is bigger than this one [pointed to angle BDC]. We know that this angle equals this one [marked angles ADB & DBC congruent] by alternate interior angles. So, then we see that this part [pointed pen at angle ADC] does not get bisected since this angle is larger than this angle [wrote inequality].
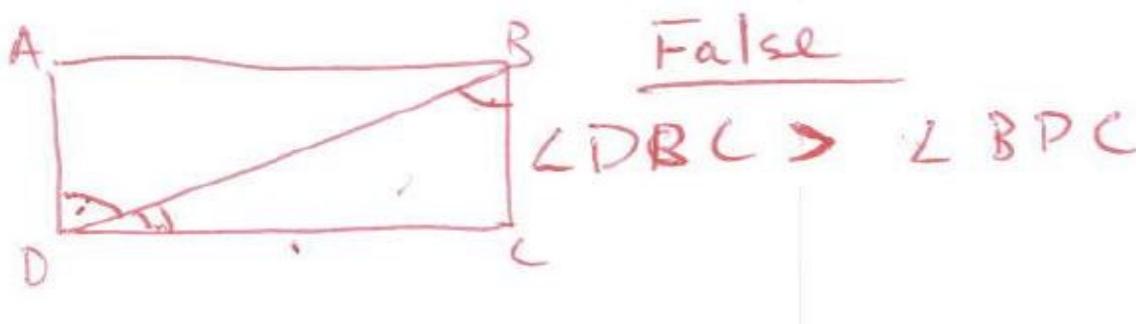

Figure 1. Mike's Figure Used to Refute Conjecture 2

Although Mike's argument has several unjustified statements which make it not a complete refutation, the gaps in his logic are rather minor. Mike's argument began by proving that the measure of angle DBC was greater than the measure of angle BDC. He likely used the fact that if one side of a triangle is longer than another side, then the angle opposite the longer side will have a greater measure than the angle opposite the shorter side. He then used the fact that the sides of a rectangle are parallel and that alternate interior angles created by parallel lines are congruent to equate the measures of angles DBC and ADB. His argument was likely finished using substitution to write the inequality in Figure 1 which proved the angles were not bisected. Even though Mike did

not provide these reasons, it seems difficult to imagine him creating his argument without having considered them.

Mike finished his work by saying, "And I only got this [pointing to his RAA argument] from that idea [pointing to his conjecture] because I knew that these [motioning fingers across angles ADB & BDC] were going to be 45-degree angles and that just popped in my head." Creating the amended conjecture helped Mike realize why the original conjecture was false, which in turn helped him to create his RAA argument. It was clear that throughout his work on the conjectures Mike's work was motivated by his desire to understand why the conjectures were true or false and he used the different subprocess of reasoning to inform the others. Interestingly, Mike's conjecture is over specialized because he trades rectangle for square, but after completing the proof he was able to correctly generalize the conjecture to rhombi, the largest class of figures that makes the conjecture true. Thus, he demonstrated how all aspects of reasoning can connect to each other.

In total, five PSTs created a formal argument to refute at least one conjecture. Mike was the only PST to create more than one and he created two. Two other arguments were considered informal refutations. These arguments were similar to Mike's in that they included statements that should have been justified but were not. The difference was in the degree of sophistication required to make that justification. As was said before, it is hard to imagine that Mike was not aware of the theorems required to justify his statements. Alana's refutation for conjecture 1 was considered informal because her unjustified statement required significant work to justify. She drew the figure in Figure 2 below and said that the triangle with sides marked "2" would have a smaller area than the triangle with sides marked "4" because the two sides were smaller, and the two triangles shared the third side. This is a valid statement, however, justifying it is not straight forward. Alana likely validated this statement using perceptual reasoning (Harel & Sowder, 1998) because this was a common form of reasoning for her. She, like other participants, is likely adding that the theorem is being applied to the particular instantiation she drew, instead of being a general statement about triangles. It is unfortunate that these types of refutations were not more common among the PSTs, but that is not meant to say that the PSTs did not attempt to create arguments of this form. In fact, another common method to refute the conjectures, referred to as faulty logic, was like Mike's, but the refutations were not valid. These are explored next.
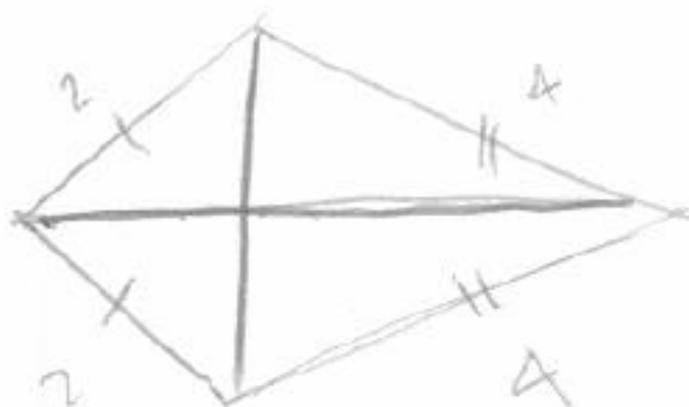

Figure 2. Alana's Figure Used to Refute Conjecture 1

Faulty Logic – Macy's Refutation for Conjecture 1

Four PSTs tried to create RAA arguments to refute conjectures but used the inverse of a true statement that was itself not true. The most used false statement to refute a conjecture was; if two triangles are not congruent, then their areas are not equal. It is likely that the PSTs were using the inverse of the true statement, if two triangles are congruent then their areas are equal. This was frequently used to refute conjecture 1, although Kay used this reason to refute conjecture 11, a true conjecture. Interestingly, all the PSTs had previously confirmed that this false inverse is in fact false on the survey. Thus, it was not a simple oversight or a lack of understanding of logic on their part.

Macy's faulty logic refutation was a bit more sophisticated than those explained above. She showed that two sides of the triangles ABC and ACD were not congruent. You can see this from her markings in Figure 3 she considered AB and CD to be congruent and AC to be a shared side. She went on to say that the other two sides [BC and AD] were not necessarily congruent. Using these two claims she said the areas could not be equal. However, this conjecture is false because it is possible for two triangles to have two pairs of congruent side lengths, while the third pair are not congruent, and still have equal areas.
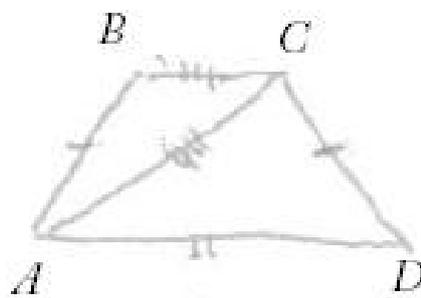

Figure 3. Macy's Figure Used to Refute Conjecture 1

Like Alana above, Macy may have added an additional condition to her claim. Namely, that the conjecture holds for instances that look like the picture she drew, an isosceles trapezoid. In fact, it would be impossible to refute her claim using her particular instantiation. However, her claim–two triangles with two pairs of congruent sides and a pair of non-congruent sides creates triangles with different areas–is not valid in a general sense. It is possible that this was simply a lapse in judgment, however, these types of claims were common in the proofs and refutations for eight of the PSTs. Namely, many of their false claims would be true with the addition of condition that the claim be made for only the instantiation or class of figure with which they were currently working.

Although these refutations were in fact false, they still were aligned with culturally accepted practices for refuting claims. Thus, it seemed that the PSTs often created arguments to refute the conjectures but failed to reflect on the validity of those arguments once they were created. Refutations, Lakatos (1976) suggested, are useful in making hidden assumptions in conjectures and proofs explicit. One difficulty with refuting claims resulted from their consistent use of claims with hidden assumptions. However, when this is coupled with the

fact that these eight often saw counterexamples and proofs as the endpoint of their reasoning it is unlikely that they will examine their proofs for hidden assumptions. Again, this speaks to the lack of interconnectedness of refutations with their work when reasoning. The two non-culturally accepted types of arguments the PSTs created are explored next.

"Not True Enough!" – Karla's Over Reliance on Familiar Criteria Refutation

Karla introduced the idea of "not true enough", which four other PSTs used, although they did not explicitly use this phrase. Primarily, Karla set out to the prove the conjectures first. When she reached a point where she could not foresee finishing the proof, she felt this was enough evidence to refute the conjecture. However, she did not appear completely convinced by her work. Her work on conjecture 2 is presented as an example. She drew Figure 4 below and then started to prove the conjecture in the following excerpt.

Data Excerpt 2: Karla Refuting Conjecture 2 by First Trying to Prove It.

Karla: So, I have a rectangle and then I want to see if this bisects this angle [traced around angle BAD]. Well, in order for that to happen, this angle and this angle would have to be congruent [marked angles DAC & BAC congruent]. So, to do that, I would try again to do congruent triangles. These are parallel since a rectangle is a parallelogram. I could do alternate interior angles, so this angle is congruent to this angle [marked angles BAC and DCA congruent]. Then, I could do the same thing with these angles [marked angles BCA & DAC congruent]. … Well, now I have two triangles [traced triangles ABC & CDA with her pen] that are congruent by angle-side-angle, where the side is the side that it shares. But I want to say that this one is false because we don't know that these two angles [pointed to angles BCA & DCA] are the same. We know that these two angles [pointed to angles BCA & CAD] are the same, but we don't know that these two [pointed to angles BCA & DCA] or these two [pointed to angles BAC & DAC] are the same. And there isn't anything in that proof that helps.
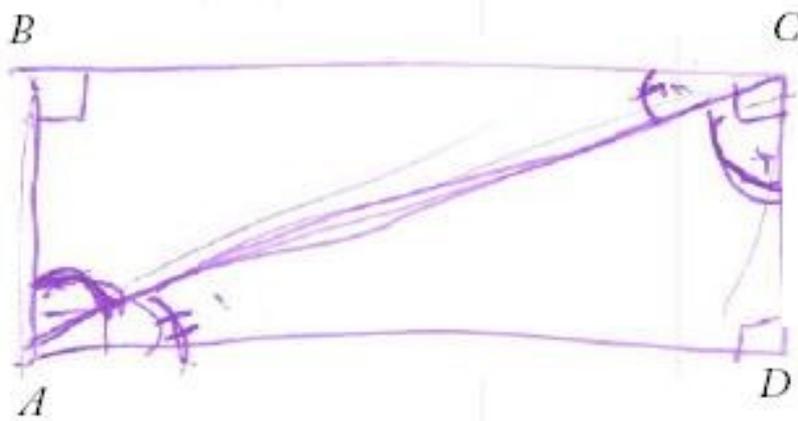

Figure 4. Karla's Figure to Prove Conjecture 2

At this point she did not seem entirely convinced the conjecture was false because of a hesitancy in her voice. She drew a much longer and skinnier rectangle with both diagonals and said, "Well it makes sense. It depends

on the size of the rectangle." Her drawing, however, was not accurately drawn which likely made it difficult for her to perceptually refute the conjecture. As I probed her thinking, we had the following exchange.

Data Excerpt 2 (Continued): Karla's Use of Not True Enough.

Interviewer: Would you say that what you've done here shows that it's not true?

Karla: I would say I couldn't prove it true.

Int.: Okay. That makes it not true then?

Karla: [Laughs] Not true enough! To me, I don't see the path where I could prove it to be true, and I can see a counterexample, in my head at least, so I would assume it's not true.

Her statement, "I would assume it's not true", clearly suggests she is still not sure that the conjecture is false. Again, her counterexample was not well drawn. When she said, "at least in my head" it seemed as if the counterexample she had drawn was not in fact a counterexample for her because she could not "see" that the angles were not equal. For the next conjecture, she similarly arrived at a point where she could not finish the proof and created a counterexample. Because her counterexample was more clearly drawn, she answered with certainty that the conjecture was false.

Karla typically began by attempting to prove the conjectures, and on multiple occasions arrived at what appeared to me as an obvious place to use RAA. For example, based on the way she marked the diagram the triangles ACD and ACB would be congruent, making segments AD and AB congruent. This is a statement she would clearly recognize as being absurd. This work not only refutes the conjecture, but it also suggests a class of figures for which the conjecture is true. Again, this would allow her to continue in the reasoning processes. This is the type of work that Weber (2009) described Isaac doing when he reasoned syntactically to refute conjectures. Although she has an example drawn, this example is merely a tool to aid her explanation and not a means for her to make sense of the conjecture. Karla then does something quite interesting; she switches to reasoning semantically by considering an extreme case of rectangle. The following exchange demonstrates what makes this change in reasoning interesting.

Data Excerpt 2 (Continued): Karla Separating Syntactic & Semantic Reasoning

Interviewer: So, do you usually do stuff like that? Make a really long rectangle if you're trying to prove stuff?

Karla: No, if I'm trying to think of a counterexample in my head. Basically, I got to the point in the proof where I thought, okay, I don't think I can prove this to be true. So, then it's like okay, does that make sense? Can I make a counterexample?

Int.: Oh, so when you think of counterexamples you think like this?

Karla: Yeah, I'll think like, usually its special cases when it's extremely long. Like with triangles covering all my bases an obtuse triangle, an acute triangle and like making sure it works for all of them.

She is clear that she does not reason semantically when proving but does so while refuting, which again speaks to there being a break in the two sub-processes of reasoning, refuting and proving. Moreover, it provides an answer to Weber's (2009) question as to whether it would be useful for an individual to be able to harness both

semantic and syntactic reasoning in their problem-solving processes. Clearly, Karla can do both, but shockingly she does them for different purposes. Even more interesting is that it would have been Karla's syntactic reasoning that would have helped her finish her refutation because it was difficult for her to create a valid counterexample that visually refuted the claim. Finally, Karla's work provides a potential reason for using this type of refutation, which Potari and colleagues (2009) found to be common among their participants. Namely, because the reasoning processes for proof and refutation are distinct and proof is over emphasized in mathematics, the participants overly focused on creating proofs. Five other PSTs started their work like Karla for each conjecture by first trying to prove it, a task that seems far more daunting than examining examples. In the discussion, I explore how Karla's work is more sophisticated than her simply stating that because she cannot write a proof the statement is false. Next, I move to the final method of refutation used by the PSTs.

"That's Only True When ..." – Kay's Refutation by Providing an Alternative Conjecture

Kay exclusively used this type of reasoning to refute conjectures. Instead of stating that the conjecture was false for a given reason, Kay preferred to state the conjecture was false because it would be true for a more specialized class of figures. For example, to start her work on conjecture 5, Kay said, "Okay, I'm going to do five and that's false! Because that's [pointed to the conjecture] only true when it's an isosceles triangle [drew triangle ABC and marked sides AB and AC congruent] and you take the median of this side [pointed to side AC]."
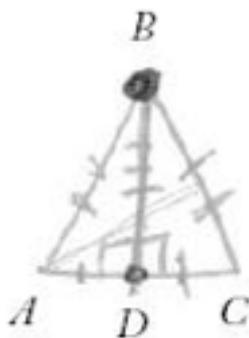

Figure 5. Kay's Example for Her Amended Conjecture 5

In this case, Kay noted a class of figures that does make the conjecture true, and it is also the largest class of figures that would make the conjecture true. When reasoning this way, six of the twelve instances the PSTs suggested a conjecture that was true for the largest class of figures. In three cases, the PSTs under specialized the class of figures that made the conjecture true. Meaning that they did not add sufficient conditions to make the conjecture true. Lucinda, for example, amended conjecture five by adding the condition that the triangle be isosceles, but failed to add the condition that only one median of an isosceles triangle creates two congruent triangles. In three other cases the PSTs over specialized the conjecture like Mike did above by adding conditions that were unnecessary.

Although this method was used only 25 percent of the time, it was used at least once by eight of the thirteen PSTs. Again, this could have been brought on by the structure of the task, but the regularity of it suggests a lack

of familiarity with how to refute a conjecture. As for a potential reason for this type of refutation, these eight PSTs were more concerned with establishing if a conjecture was true or false versus making sense of the conjecture. Lucinda for example was explained the goal of the interview, but started by writing, "#1 True, #2 True, #3" before I stopped to ask if she would do the rest of the question, or in the case of conjecture 1 provide a proof. It was common for the PSTs to not be concerned with determining why the conjecture was in fact false. These same eight PSTs preferred to create specific counterexamples and, like Sadie, considered this to be the endpoint in their work. Whereas a true conjecture was a point where they could move forward in their reasoning process, namely write a proof. Another possible suggestion for why the PSTs use these types of refutations is that their teachers refuted their conjectures this way. These types of refutations were commonly used by the teacher in Sekiguchi's (1991) study, and Knuth (2002) claimed that the in-service teachers in his study over emphasized the role of verification for proof. Thus, it seems quite plausible that the PSTs have developed a sense that it is more important in mathematics classes to prove conjectures versus refute them. Meaning, their emphasis on stating a valid conjecture versus providing a refutation for the conjecture may be how they interpret the work they are supposed to do in this situation.

Discussion and Conclusions

This section contains a summary of the difficulties the PSTs faced when refuting geometric conjectures. Namely that there was a disconnection between refutations and other subprocesses of reasoning and there was a disconnect between syntactic and semantic reasoning. Ways of addressing these issues are discussed throughout.

Connecting Refutations to Reasoning - Addressing Over Reliance on Proof and Specific Counterexamples

These PSTs demonstrated that they could create refutations that went beyond specific counterexamples. This is positive because there is the potential that they will use these power pedagogical counterexamples in their teaching practice (Peled & Zaslavsky, 1997). Additionally, it is an advancement over the participants in Potari and colleagues' (2009) study. However, the arguments they created were often not mathematically viable and specific counterexamples were still common. These PSTs seemed to not have experienced the inter-related process of reasoning or at the very least were unaware of the powerful role that refutations can have on their reasoning.

To make a stronger claim, it seemed that the participants' typical experience with reasoning involved proving claims versus refuting them. If they were asked to refute conjectures, this likely signaled the end of their reasoning process. I make this claim for four reasons. First, they saw counterexamples as an end to their reasoning process. Second, when they were asked to continue in the reasoning process by amending the conjectures, they did not use what they had learned from their refutation to create a new conjecture. Third, they preferred to work on true conjectures, even going so far as to regularly refute claims by simply providing an alternative conjecture to prove. Finally, students like Karla demonstrated that they were capable reasoning both syntactically and semantically but were not always able or willing to harness those ways of reasoning in all the

subprocesses that make up mathematical reasoning. Each of these are explored more next along with a discussion of ways to improve pre-service teachers' reasoning experiences.

Specific counterexamples were common, and this was largely because the PSTs saw the creation of counterexamples as the end to their reasoning. One counterexample is sufficient to refute a conjecture and seeing this counterexample as an endpoint in many ways is fine mathematically. However, the PSTs might not be prepared to build classroom environments that support the inter-related aspects of reasoning described by Lannin and colleagues (2011). There are mathematical consequences as well. Many PSTs clearly separated the work they did to refute conjectures from the work they did while amending the conjectures. Many scholars (Lakatos, 1976; Lannin et al, 2011) consider these to be mutually informing practices. Because these PSTs saw the counterexample as an end, they often amended the false conjectures by moving along hierarchies they had for triangles and quadrilaterals. This caused their conjectures to be invalid or non-economical. Of the thirty conjectures that the PSTs created six of them were invalid and ten were not economical. Of the remaining conjectures they created three were incoherent or lacked enough specificity to be a mathematical statement (e.g., some quadrilaterals have congruent diagonals). Additionally, when the PSTs refuted by providing an alternative conjecture, creating valid conjectures was difficult (only 6 of the 12 instances were valid). If their refutations were more connected, they might use their work during the refutation to make sense of a particular property that they felt the figure needed to make the conjecture true like Mike did.

Another issue that stymied the PSTs' refutations was that they seemed to prefer valid conjectures. All the PSTs completed at least one true conjecture despite there being more than twice as many false conjectures. It seemed the PSTs were avoiding the false conjectures. The reasoning pattern "providing an alternative conjecture" seems to be a consequence of this preference. Instead of truly refuting the conjecture, the PSTs often offered a new conjecture to prove. This is problematic from a teaching standpoint because these refutations could result in their future students developing a belief that mathematics is exclusively about proving conjectures–a belief that these PSTs seemed to hold. Moreover, it will be difficult for them to properly refute their students' conjectures because the alternate conjectures they suggested were often invalid and uneconomical. Additionally, their methods of refutation were rarely formal. They did create a valid counterexample 85% of the time, but were only able to explain that counterexample, at least informally 19% of the time. Counterexamples refute conjectures, but teachers ought to explain their counterexamples.

Before discussing how to address these issues, it is important to note that if by reasoning we, as mathematics teachers of all levels, only mean to verify something is true, then we are doing our students a disservice. We are not allowing them to see the powerful role that refutations can play in their reasoning. Particularly without exploring why they felt a conjecture was false, the PSTs were prone to over or under specialize their conjectures, but also, they were not allowed to experience a contradiction and in turn potentially learn (Balacheff, 1991; Piaget, 1971). Or as Borero and colleagues (1996) suggest, this type of work breaks the cognitive unity between the conjecture and the proof. On conjecture 11 for example, Sabrina seemed genuinely surprised the conjecture was true after trying to explain that it was false. Her refutation helped her see the validity of the conjecture. Although we might see proving a conjecture true as the only opportunity for growth

and learning, it is not, and in fact refutations, despite resulting in a negative outcome, are powerful learning opportunities.

To address the issue, we should allow students to participate fully in the reasoning process. This means there should be some level of uncertainty about the truth of statements among the students. Yopp (2013) was able to discuss statements without inadvertently passing authoritative judgement on their veracity by using the terms conjecture, proposition, and theorem. A conjecture in his class was a statement they did not know the truth value of. A proposition was a statement that was assumed to be true, and theorem was a statement that was proven to be true. The conjectures the PSTs were asked to evaluate in this study were created by fictional students, so they seemed more willing to be skeptical. Simple changes like these can open the task up to include both refutation and proof as a viable option. It seems almost obvious that students would learn that proof and refutation are mutually exclusive options when they are only asked questions that ask them to prove true statements or find counterexamples for false statements.

Connecting Semantic and Syntactic Reasoning

The reasoning pattern "over reliance on familiar criteria" which other studies on refutations have also found to be common (Giannakoulias, et al., 2010; Potari, et al., 2009) seems less negative as Karla's work demonstrates the potential sophistication required to produce such a result. She began by trying to prove the conjecture but arrived at point where none of the known triangle congruence conditions could be met. In Potari and colleague's (2009) study, the participants appeared to stop here and claim the statement was false, making their thinking appear to be procedural in nature. However, that may be an unfair characterization given that the participants were not given a chance to explain their reasoning like Karla was. If in fact the participants were not able to continue, then as Potari and colleagues suggested, this is a lack of syntactic reasoning. Therefore, one potential way to improve PSTs' understanding of refutations would be to fill in this gap in their understanding of logic. More specifically, it would be helpful to provide PSTs with situations where conjectures are true, but there is not enough information to prove them by standard means. This may help them give up their procedural understanding that a lack of proof warrants a statement being false. Additionally, it would be helpful to have students be explicit about their thinking when providing this type of refutation. This would provide them the opportunity to explain their thinking if possible. If syntactic reasoning is not useful in this endeavor, then it might be helpful to suggest that the student examine examples like Karla did.

Karla went further in her reasoning by examining examples, and she did so in a structured way. She examined extreme cases and each case of subclass for a figure. This is a form of semantic reasoning. Being able to reason both syntactically and semantically seems very powerful, but Karla used different types of reasoning during different parts of the reasoning processes instead of harnessing the two types. She may not be aware that reasoning semantically can be productive (or permissible) when proving or that reasoning syntactically can be productive when refuting. Again, finding a counterexample, semantic reasoning, seems so synonymous with refuting that RAA arguments may not be considered an option. It was common for the PSTs to express that providing examples was not permissible while proving (Creager, 2016). Certainly, examples cannot prove a

general statement, but avoiding thinking about examples is contrary to what Harel and Sowder (1998) described as transformational reasoning, a critical aspect of deductive reasoning.

To connect syntactic and semantic reasoning we need to be explicit about when they are being used and what value there is in using the two types of reasoning. In Mike's case, we can clearly see that by considering the need for the angles to be 45 degrees, syntactic reasoning, he was able to formulate a valid conjecture. It seems likely that Karla could have been brought to understand a refutation like Mike's. Moreover, she might then see the value of reasoning syntactically to refute conjectures after being presented with a refutation like Mike's. Finally, reasoning semantically like Karla did by examining extreme cases and being aware of what it meant to be extreme, in her case "really long sides", is informative because one might realize that the adjacent sides need to be equal in length.

Yopp (2013) similarly found success by asking students to reflect on how they found their counterexample and to think about what their counterexample told them about why the statement is false. By asking them to evaluate how they created their counterexamples, the students are building metacognitive skills, but it also forces them to be explicit about their thinking. This provides an opportunity for the teacher to gather evidence as to whether students are reasoning syntactically or semantically and provides an opportunity to have them consider whether reasoning the other way would be helpful. Yopp (2013) provided clear suggestions for syntactical ways to support students as they explore ways to amend the conjecture. However, there are semantic ways to support their reasoning as well. The participants in this study moved along hierarchies of figures they knew. This search process was unproductive. Instead, the students might be asked to examine properties that caused the conjecture to be true or false, like Mike did. Or, they could be asked to examine extreme cases, or to examine an example of each subclass of figure like Karla did. Examining examples in a structured way is a critical skill in the transition from empirical to deductive reasoning (Harel & Sowder, 1998). Thus, refutations may serve to help PSTs learn to prove. An additional suggestion would be to have students reflect on their proofs and refutations. This will help make proof part of the cyclical process and help students to develop skills with proof evaluation (Selden & Selden, 2003). As students are examining steps in their argument, have them consider a step for a diversity of figures. This will help connect syntactic and semantic reasoning of reasoning and develop what Harel and Sowder (1998) referred to as transformational reasoning.

These suggestions are for PSTs, and they may not be useful for younger students as younger students may not have the ability to follow the logic of a syntactic argument. However, refutations play a critical role in doing and learning mathematics and there is more to refutations than simply creating counterexamples. Hopefully, this article has shed some light on the diversity of things one might do when refuting and ways of improving PSTs' understanding of refutations.

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