www.ijemst.net

Methods of Explicitly Teaching Generalization in the Mathematics Classroom and Indicators of Success: A Systematic Review

Jay L. Jackson
University of North Alabama, United States

Cynthia L. Stenger
University of North Alabama, United States

To cite this article:

Jackson, J.L. & Stenger, C.L. (2024). Methods of explicitly teaching generalization in the mathematics classroom and indicators of success: A systematic review. *International Journal of Education in Mathematics, Science, and Technology (IJEMST), 12*(4), 1109-1126. https://doi.org/10.46328/ijemst.4171

International Journal of Education in Mathematics, Science and Technology

2024, Vol. 12, No. 4, 1109-1126 https://doi.org/10.46328/ijemst.4171

Methods of Explicitly Teaching Generalization in the Mathematics Classroom and Indicators of Success: A Systematic Review

Jay L. Jackson, Cynthia L. Stenger

Article Info

Article History

Received:
29 December 2023

Accepted:
19 June 2024

Keywords

Generalization
Systematic review
Mathematics education
Teaching methods

Abstract

Arguably, the ability to generalize is an essential characteristic of students who succeed in mathematics and computer science. Teachers should be aware of its presence in their classroom and have tools to encourage its existence. Thus, it is important to examine the successful ways that generalization is taught. This systematic review examined and synthesized literature regarding teaching generalization and looked for indicators of success when teaching it. The findings showed that the most common way generalization is taught is through patterning tasks, primarily in K-6. Successful generalization was most frequently assessed by "caught not taught" methods. The primary indicator of success was an algebraic expression which, unfortunately, does not guarantee that generalization has occurred in the mind of the student. In summary, very few methods of teaching generalization have actually been journaled, and there is a dearth of methods to evaluate or assess if generalization has occurred.

Introduction

Among mathematics educators, generalization is often noted as being an integral component of the foundation for students who are successful in the subject. Mason (1996) describes generalization as "the heartbeat of mathematics" (p.65), adding that if teachers are not aware of its presence in the classroom and are not encouraging students to work at articulating their own general expressions, then mathematical thinking is not happening in that classroom. Ellis et al. (2017) state that generalizing is fundamental to mathematical activity and serves as the means to construct new knowledge, while Carpenter and Franke (2001) note that "when students make generalizations about properties of numbers or operations, they make explicit their mathematical thinking" (p. 156). The National Council of Teachers of Mathematics' (NCTM) (2000) publication, *Principles and Standards for School Mathematics*, summarizes the importance of generalization by stating that people who reason and think analytically are ones who can note structure, patterns, and regularities in various situations. Further, the patterns and structure are an impetus for making and investigating conjectures. In NCTM's (2014) publication *Principles to Actions*, productive beliefs about teaching include that students should be knowledgeable of general methods along with standard algorithms and procedures in their problem solving. Generalization is certainly recognized by prominent mathematics education researchers as having great importance, making it somewhat unusual that is defined in so many different ways.

Ellis (2007b) describes generalizing actions and reflection generalizations. She asserts there are three major generalizing action categories to include *relating*, *searching*, and *extending* (i.e., what students do when they generalize). *Relating* occurs when the learner forms an association between two or more problems; *searching* is repeating an action to try to find some element of similarity; *extending* involves building on a pattern or relation to build a more general structure. Reflection generalization encompasses making identifications or statements and formulating definitions. She further notes that developing a categorization system for these generalizations allows researchers to understand how students construct rules for making their own generalizations, noting that these processes are not well understood (Ellis, 2011), thus eliciting the need for continued research. In proposing a formal definition, Ellis et al. (2017) use the term *generalizing* to reference processes of identifying commonality across cases, extending one's reasoning beyond the range in which it originated, and/or deriving broader results from particular cases. Additionally, the term *generalization* refers to the outcome of these particular processes.

Harel and Tall (1991) distinguished three definitions of generalization, making them dependent on the manner in which individuals mentally construct a concept and based on schemas. The first, expansive generalization, occurs when the learner takes an existing schema and extends its range of applicability without reconstructing it. Reconstructive generalization, their second definition, is similar to the first definition except that the existing schema is reconstructed in order to widen the range of its applicability. Here, a previously used schema is changed and enhanced before being added to a more general schema. The third definition, disjunctive generalization, occurs when a learner moves from one context to another and, in the process, constructs a completely new schema to deal with the new context. A more simple way to summarize these definitions is that it enlarges one's range of thinking beyond a case that is currently being considered or, in the words of Harel and Tall (1991), "applying a given argument in a broader context" (p. 38).

A third researcher, Mason (1996), describes generalization as one way to broaden the scope of reference and apply a result. One can then expand that concept to other contexts by lessening restrictions and seeking to make connections to other topics and processes. Simply speaking, Mason et al. (2010) define generalization as the process of moving from observing a few instances to extrapolating a wide class of cases. This process starts when one sees an underlying pattern, even if he or she is unable to articulate it. Mason et al. (2010) summarize, "Whereas specific results may in themselves be useful, the characteristically mathematical result is the general one" (p. 8).

These and other definitions of and ideas about generalization (Davydov, 1990; Dörfler, 1991; Martino & Maher, 1999; Mata-Pereira & da Ponte, 2017; Mitchelmore, 2002) abound in mathematics education. The definition of generalization used by researchers often depends on the group of people participating in the research. Consequently, the method for teaching generalization in a mathematics classroom is also diverse. In this review, the aim is to examine and synthesize the literature about *teaching* generalization, no matter how it is defined. By synthesizing the literature on this topic, mathematics teachers may find a particular method that seems to be successful for teaching generalization (however it is defined) or find that no clear method exists for doing so. The motivation for undertaking this systematic review was participation by the authors in a group that uses computer programming in an attempt to explicitly teach generalization in the classroom. This method, other than those that simply look for patterns in geometry or number sequences, may be the only instructional model that makes such an explicit attempt to teach generalization. For this reason, papers that have been published by this group were not included in the review.

A systematic review, in lieu of a meta-analysis, was chosen for this paper because it has clearly stated objectives with pre-defined eligibility criteria for the included studies. This type of review has specific and reproducible methodology, uses a systematic search that endeavors to identify all applicable studies to be used, and should produce a systematic synthesis and report of the characteristics and findings of the included studies (Green et al., 2008). A meta-analysis was considered but did not seem consistent with the objectives of answering the research questions.

Method

The purpose of this study is to synthesize research performed by instructors teaching in the classroom where at least one of their desired outcomes was generalization, however it is defined. The research questions guiding this review are as follows:

1. What methods are used to explicitly teach generalization in a mathematics classroom setting?
2. What indicators of success have mathematics teachers observed when attempts to explicitly teach generalization are undertaken in their classroom?

The authors used a systematic review methodology (Hannes et al., 2007; Strech & Sofaer, 2012; Wright et al., 2007) in pursuit of finding articles related to the goal of this study. This methodology follows a four-step process for writing a review, including: (a) formulating a review question(s) and eligibility criteria, (b) identifying all of the literature that meets the eligibility criteria, (c) extracting and synthesizing data, and (d) deriving and presenting results (which should answer the review questions). After consulting with a pair of research librarians concerning the research questions, two databases were chosen to use in conducting the search for relevant literature: ERIC and APA PsycInfo. A simultaneous search using these two databases was conducted on December 4, 2020, for the following terms appearing anywhere in the articles: (teach* OR instruct*) AND (math*) AND (generali*). Also included were the following limiters: (a) publication dates ranged from 1950 to present, and (b) the publication was a scholarly peer-reviewed journal. After a first run and seeing quite a few listings involving autism and disabilities, the authors chose to make another run including the limiter "NOT (autism or disab*)," and this final search generated a total of 1626 non-duplicate publications.

Next, the authors used the following inclusion criteria in deciding which articles would be included in the analysis: (a) explicit teaching of generalization (or at least an attempt to) was actually taking place as part of the research, and (b) the participants were learners of mathematics. These criteria were used in the next phases of the screening process: a title screen, an abstract screen, and a full-text screen. After screening the title of each of the 1626 hits in the original search, 179 titles remained. The primary reason for exclusions in the title screen were that it was obvious that explicit teaching of generalization was not the focus of the study. Secondarily, the researchers were simply evaluating lessons that had been taught to see if some generalization had occurred rather than an explicit intervention for teaching generalization. Of these, 131 were discarded after the abstract screen, leaving 48. After scanning the full text of the remaining articles, 30 met the inclusion criteria. Further, to locate studies which may have been missed in the initial search, an ancestral search was conducted on each of the remaining articles to look for referenced articles that should possibly have been included. This search yielded only one additional article, leaving a total of 31 articles that were used in the review. Because of a global pandemic, the intended report was delayed. Consequently, on November 19, 2022, an additional search was conducted using the same limiters while extending the dates to present. This produced an additional 106 articles, 77 of which were eliminated in the title screen, with 17 more eliminated in the abstract screen and 6 more in the full-text screen. A PRISMA diagram (Moher et al., 2015), accounting for the total of both search and inclusion processes, is detailed in Figure 1.

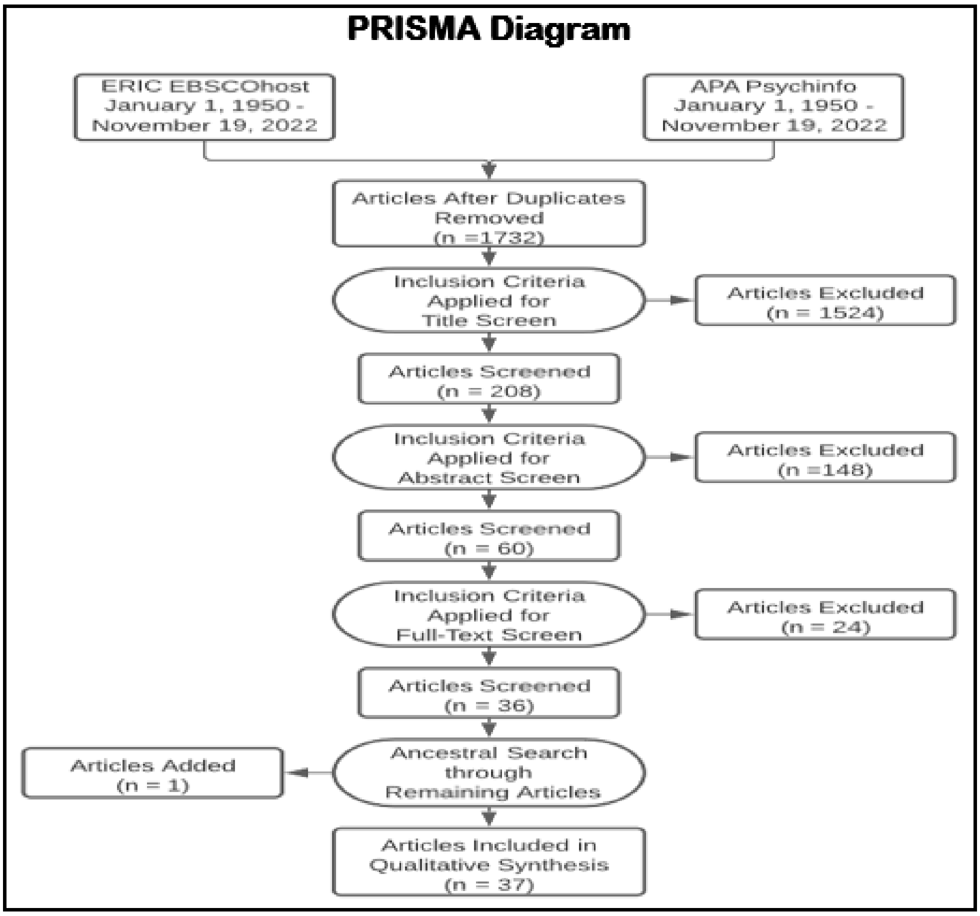

Figure 1. PRISMA Diagram

With the 37 articles in hand, the analysis phase of the systematic review began. First, the authors read each of the articles with the purpose of documenting summaries of each methodology, findings, and conclusions. Particularly, each methodology was examined for instructional techniques being implemented and for any indicators of success in the findings. Employing a thematic analysis approach (Braun & Clarke, 2012), summaries of the articles were used to code each article so that data could be linked to broader theoretical and conceptual issues while answering the research questions regarding methods for teaching generalization in the classroom and indicators of success in doing so. The process included six phases: familiarizing oneself with the data, generating initial codes, searching for themes, reviewing potential themes, defining and naming themes, and producing the report (Braun & Clarke, 2012).

Results and Discussion

Each article was read for the purpose of generating the initial codes. These codes included grade level, sample size, duration of the study, methods of teaching, and indicators of success. These and any clarifying categorizations within the code are shown in Table 1.

Table 1. Initial Codes

Code	Categories
Grade level of participants	1 = PreK-1, 2 = 2-5, 3 = 6-8, 4 = 9-12, 5 = college students, 6 = in-service teachers
Sample size	1 = 20 or less, 2 = 20-60, 3 = 60-100, 4 = more than 100
Duration of study	1 = 1 week or less, 2 = 1 month or less, 3 = 1 semester or less, 4 = 1 school year or less, 5 = greater than a school year, 6 = unk
Methods of teaching	categorization not necessary
Indicators of success	categorization not necessary

After this initial coding, the process of searching for themes began, leading to a review of potential themes, and eventually naming and defining the themes (Braun & Clarke, 2012). From the outset under the methods of teaching, it was clear that patterning tasks was the dominant theme, located in 21 of the 37 articles. The articles I chose revealed 1) specific patterning tasks while teaching and 2) other than patterning tasks while teaching, as the two themes for this study. From here, sub-themes developed within each of these principle themes which will be discussed later in the findings. Under the indicators of success, three major themes developed: determining a formula, describing a pattern with words, and answering math facts. Themes and sub-themes for each of these two codes are shown in Table 2.

Table 2. Themes and Sub-Themes of Initial Codes

Code	Themes	Sub-Themes
Methods of teaching	Using patterning tasks	Figural patterns
		Physical patterning
		Number patterns
	Using other than patterning tasks	Direct feedback
		Manipulatives
		Technology
		Miscellaneous
Indicators of Success	Determining a formula	
	Describing a pattern with words	
	Answering math facts	
	Miscellaneous	

Preceding the discussion of thematic findings, some general aspects of the study are shared in tabular form to include grade level of the learners, sample size, and the duration of each of the studies. This information is provided so that the reader may have a sense of how this type of research has been pursued by others. After these tables of contextual data, the thematic findings are presented, followed by a brief discussion on their implications for instructional practices and possible future research going forward.

Contextual Aspects of the Literature

The following tables display statistical findings from the contextual features of the topics listed in the initial coding. Table 3 shows that most of the studies were conducted with students in the 2nd through 5th grades. Interestingly, only two teaching experiments (5%) were affected at the high school level. One study, performed in Mexico, reported using in-service teachers. The number of participants in each study is shown in Table 4. Most of the sample sizes reported had participants numbering less than 20 learners. Table 5 reveals that most of the studies were accomplished in less than one month, although that data was not reported in six of the studies observed. Longitudinal studies lasting more than one year were described in six of the articles.

Table 3. Grade Level of Participants in the Studies

Grade level of participants	Percentage of publications
PreK to 1	8% (3/37)
2 to 5	38% (14/37)
6 to 8	27% (10/37)
9 to 12	5% (2/37)
College	19% (7/37)
In-service teachers	3% (1/37)

Table 4. Sample Size in the Studies

Sample size	Percentage of publications
20 or less	38% (14/37)
20 to 60	32% (12/37)
60 to 100	11% (4/37)
More than 100	19% (7/37)

Table 5. Duration of the Studies

Duration of the studies	Percentage of publications
One week or less	22% (8/37)
More than one week and less than one month	22% (8/37)
More than one month and less than one semester	11% (4/37)
More than one semester and less than one school year	13% (5/37)
Greater than a school year	16% (6/37)
Unknown	16% (6/37)

Teaching While Using Patterning Tasks

In thematizing the literature, three sub-themes related to teaching with patterning tasks emerged. These include teaching with figural patterns, physical patterning, and number patterning. For the purpose of this research, physical patterning is defined as using manipulatives to create patterns or actually drawing out a patterned representation by hand.

Yao (2022) says that "pattern generalization is a typical generalization in school mathematics. Pattern generalization tasks can be classified broadly as numerical when the pattern is listed as a sequence of numbers, or as figural when the pattern is set in a pictorial context showing one or more configurations" (p. 2). Since over half of the articles explain using patterning in some fashion as the mode of instruction, this shows it to be the preferred method of teaching students to generalize. Most patterning tasks have as an end goal to direct the learners to generalize a pattern in such a way that the prediction of later entries into a pattern sequence can be represented with a formula or verbal description. Of the 21 articles that contained themes with patterning tasks, eight were sub-themed as figural patterns, nine as physical patterning, and four as number patterns. Following are the thematic findings that provide further insight into teaching while using patterning tasks.

Figural Patterns

For these lessons taught in grades 3-5, the figural patterns were all growing patterns. As an example, one teaching task involves the number of people who can be seated around one or more square desks. The task is two-fold: in the first case, guests can only sit across from each other at the desks; and, in the second case, an additional guest can sit at each end of the desk arrangement (Stephens et al., 2016). Desks are added, and the student must predict how many guests can be seated with, for example, 100 desks. Then, the students try to come up with a rule to describe the growing pattern, in general. This rule is considered to be the generalization of how to make this prediction. During this type of patterning task, the teacher guides the learners in noticing the growing pattern, adding to the pattern, probing the nature of the relationship, and then predicting and generalizing (Hourigan & Leavy, 2015).

At the 8th grade level, Rivera and Becker (2008) designed tasks calling for figural and algebraic generalizations. In this task, adjacent squares are shown (or built with "toothpicks), and the learner is asked to find a rule that will predict how many total toothpicks are needed to finish with a given number of squares. Another task, called the W-Dot Sequence Problem, requires the student to use circles to form a "W" shape, adding an extra "dot" to each of the four "legs" of the W each time. Again, the students must then find a rule that will tell how many dots would be in some extended iteration of the "W" figure. Instruction given before these tasks had as its goal to provide students with opportunities for problem-solving where generalization was required.

Yao (2022) used a task-based interview as the teaching method in a generalization study with preservice secondary mathematics teachers. Two tasks were assigned involving the expansion of geometric shapes, namely a triangle, pentagon, and rectangle. As an example, using the former two shapes, dots were located in the array at the vertices of each successive expansion. The goal was to determine the number of total dots when the figure had particular side lengths. In this task, the instructor would engage with the students, asking them to reflect on what they were doing or share what they were thinking as they worked to complete each task.

The same types of figural patterns were also predominantly used in teaching a select group of middle school mathematics teachers (Moguel et al., 2019). The main difference in these patterns is that the middle school teachers are expected to make their predictions using algebraic expressions, and the expressions are not necessarily linear. One task described necessarily involves more complicated growth patterns. In another "toothpick task," these in-service teachers must build a staircase using toothpicks. After the first four-sided stair, they must add the next step by doubling the height of the first step. Then, the third step is added by tripling the height of the first step. In each case, the height must be completed with a complete "block" of toothpicks (Moguel et al., 2019). When incorporating this particular task in the classroom, the "stairs" are sometimes just shown, and sometimes the participants are actually constructing the stairs with manipulatives. This type of problem produces a generalization that involves a quadratic expression rather than a linear one. The teaching aspect in this example was fairly simple; the task was assigned, and the participants were asked to show and justify their reasoning behind the solution they gave.

Physical Patterning

Teachers employing physical patterning in their teaching of generalization also use pattern-eliciting tasks. However, in this case, the learners generate the patterns with manipulatives, pencil and paper, or even sometimes their bodily movement. In studies conducted with Pre-K and first graders, Mulligan et al. (2020) and Mulligan et al. (2008) use the PASMAP (Pattern and Structure Mathematics Awareness Program) pedagogical model. This model, in essence, is an intervention that encourages instructors to enrich and expand students' patterning and structural thinking. One example of a PASMAP component is symmetry and formations, where students may be asked to "take one step forward and turn right, repeat four times, what shape can I make?" (Mulligan et al., 2020, p. 666). Another example has participants folding shapes into quarters in different ways and then drawing their fold lines from memory. The teaching takes in to account mathematical ideas from modeling and representing, visualizing and generalizing, and reinforcement.

Examples from grades 2 – 5 include a tile problem designed by Pinto and Cañadas (2021). Although this problem could be classified as a figural problem, some of the participants were given colored papers which allowed them to actually reproduce the picture of the tiles required to cover a floor in a particular pattern. The researchers conducted four Classroom Teaching Experiments before assigning the task. They explored how the students generalize with respect to functional relationships. During these sessions, the researchers worked with the students to help them answer questions when they were working on particular problems, with the end goal of generalizing the topic of study. Generalizations could be in the form of language, pictures, numerical quantities, algebraic expressions, or tables and graphs. The previously mentioned "W" pattern (Rivera & Becker, 2008) was used in a 5th grade classroom by Aspari et al. (2020), but in a different context, framing it as dancers in formation. In this task, it was used as the beginning of a dance formation which would develop into a flash mob. This "mob" started as a dance formation in a "V" shape. Different patterns grew in the shape of the "V" or the "W," and the students were directed to draw the different additions, creating the "mob." Here, the teaching strategy allowed the students to work in groups while answering a series of questions about the tasks with the teacher offering limited help along the way.

Middle schoolers used physical objects such as cubes to model patterning tasks about cube stickers, theater seating, sharing a pizza, and a phone cost problem (Lannin et al., 2006). The instruction lasted ten consecutive days, with the instructor and an observer planning the next day's lesson based on their conceptualization of students' thinking on that day. The classroom lessons were designed to look at students' understanding of how to generalize problems, with the tasks interspersed throughout, and with the goal of students' developing recursive or explicit rules for generalization. Rivera and Becker (2016) used semi-free patterning tasks to foster generalization in a middle school Algebra I class during a two-week teaching experiment. Instruction during the two weeks included lecture and activities that focused on linear pattern generalizations. The physical patterning tasks were given as homework problems and consisted of an initial pattern (e.g., three circles arranged as a mini-pyramid). The students were to build on the pattern in any way they wanted, making it consistently larger with each addition of however many circles they chose. Ultimately, they determined the total of circles at each stage with a function-based generalization.

In a high school setting, Pauletti and Zaslavsky (2018) used tiles to build criteria-based designs. The teaching strategy employed was task-based clinical interviews, with the researchers asking students to justify their answers and thinking along the way. Working in groups was essential in this study, as the investigation examined whether group discussions and peer interaction/collaboration helped in the understanding of generalization.

Preservice elementary teachers used pattern block squares to investigate the perimeter of a pattern block train in a mathematics methods course (Richardson et al., 2009). This study was designed to reveal how pattern-finding tasks foster the learning of generalization. Instructors examined these undergraduates' use of representations, their algebraic reasoning, and overall learning, in order to plan follow-on lessons over the course of the three-week study. A similar strategy was employed to investigate the perimeters of various polygon trains by participants who were recruited through a large state university's undergraduate psychology pool (Hallinen et al., 2021). Students were given initial patterns but sketched their own extensions. Trains were created using triangles, squares, pentagons, and hexagons.

Number Patterns

Four studies using number patterns to teach generalization were found in the search of the literature. One occurred as part of a long-term classroom-based intervention in northern Italy (Ferrara & Sinclair, 2016). This particular lesson was carried out in two sessions of four hours each in a third-grade classroom. The premise of this study was that pattern generalization is one way to introduce young students to algebraic thinking. The teacher used toilet paper to write numbers on individual sheets, followed by the use of sticky notes to label various positions of numbers; e.g., starting at 6 with the first sticky note and then a new note on every other number.

In 7th- and 8th-grade classrooms, Ellis (2007b, 2009) used tables to post number relations, sometimes giving "*x*, *y*" combinations and sometimes using real-world quantities. Ellis enlisted 7th-graders because they had not yet been taught about linear functions. Instruction occurred on fifteen consecutive school days, lasting 1 ½ hours each day. Also, 30-minute discussions took place with one student at the end of each lesson. The instruction covered topics of generalization and justification, with the students being told that they would be encouraged to explain any solutions or methods of approach to the problem tasks. The second article related the same type of instruction with a class of eighth graders, although it did not emphasize or detail the instructor input into the lesson as much. In both articles, students were given tasks involving tables of numbers and asked to answer questions that identify, extend, and predict the pattern.

The "Magic V" is an open-ended task that allows students to arrange numbers in such a way that each leg of the V adds up to the same amount. In the case of a study conducted by Wiedjad et al. (2020), the numbers 1 through 5 were selected for these classes of third and fourth graders. Although number patterns are important, in this case they do not necessarily generalize to an algebraic expression. Rather, the aim is to have the students generalize the characteristics of workable number patterns. This lesson is particularly helpful in using the commutative property of addition when generalizing answers. The teaching strategy employed allowed the students to work in pairs on the task, to have discussions involving examples and conjectures, to do additional work in pairs to test a conjecture, and to justify answers during whole-class discussion.

Teaching Using Other Than Patterning Tasks

With regard to sub-themes related to teaching generalization when not using patterning tasks, three became apparent, which called for a fourth sub-theme of miscellaneous methods. The three sub-themes were teaching using direct feedback, teaching using manipulatives, and teaching using technology. Of the 16 articles under this major heading, eight were using direct feedback in some manner as the primary method of instruction, three used technology in some manner, and two related the use of manipulatives. The remaining three articles are those teaching methods that will be discussed under a miscellaneous heading.

Direct Feedback

Direct feedback on attempted problems or assignments is a popular approach for attempts at teaching generalization. It should be reiterated that definitions of generalization vary in these studies. As an example, one study noted "evidence of generalization" (Miller et al., 2011, p. 215) when second graders were able to work an addition problem on assessments that were conducted using inverse facts, such as 3 + 4 and 4 + 3. One of the more common types of feedback stems from drill and practice. This was evident in several studies, usually with students in lower grade levels (Codding et al., 2010; Miller et al., 2011; Poncy et al., 2010; Rich et al., 2017). During taped-problems intervention, as one example, students listened to a recording of a problem and then attempted to write the answer before the recording provided the correct response (Miller et al., 2011). Another example is a strategy that used incremental rehearsal, a drill intervention that incorporates essential elements of practice (Codding et al., 2010). In layman's terms, the student drills, practices what they do not answer correctly, and then drills again.

Other types of direct feedback fall under the category of teacher questioning. Two studies (Aparicio Landa et al., 2021; Martino & Mayer, 1999) emphasize the importance of teacher-led discussions in helping students to generalize. Aparicio Landa's study (2021) with ten PSMTs termed the method "reflective conversation," stating that the teacher's job consisted of "making the ideas in the conversation flow and to promote the conversational learning cycle" (p. 46). Martino et al. (1999) simply assigned a task (The Tower Problem) and then used targeting questioning to help 4th-grade students form hypotheses and make connections, while permitting them to describe their thought processes using more generalized language. A third study described this method as having "initiated the inductive process" (Sriraman and Adrian, 2004). Here, students completed four problem-solving tasks and were then led by the teacher in a reflection on the nature of how the solutions were structured.

Manipulatives

Two studies used manipulatives in trying to help students generalize (Hill et al., 2015; Laski et al., 2021). Although manipulatives were also used in many of the lessons that were described earlier that used patterns, both of these studies used them to teach addition methods. Hill et al. (2015) used Unifix cubes to teach addition by 10 and subtraction by 1 to second graders. Adding mixed digit numbers was taught using a single frame, a set of tiles, or a combination of both, to first grade students (Laski et al., 2021). The frame works much like an abacus, while the tiles represented units of ten and units of one.

Technology

Allowing students to write programming code was a technique used in a second-grade study (Miller, 2019). Codes were written using Scratch, along with three coding robots, to draw shapes. Students had to program turns in order for the shapes to be successfully drawn. The instructor worked to support the students in helping them understand the code they were writing and by encouraging peer interaction in justifying their choices. The other two studies under this heading used dynamic geometry environments (DGE) to draw and explore geometric shapes. In the first, GeoGebra was used to examine and explain geometrical properties of conic sections (Fahgren & Brunström, 2014). Geometer Sketchpad was used in the second to look at properties of geometric transformations (Yao & Manouchehri, 2019).

Miscellaneous

In the remaining studies, other methods were employed to enhance generalization. One study used a lecture format in teaching students about variables (Blanton et al., 2016). If the students later used a variable in an applicable context to represent an unknown quantity, the intervention was deemed successful. Another study detailed an algebra intervention where students engage in four concept areas. The lessons used were designed to encourage students to notice patterns in these areas and to begin to generalize (Strachota, 2020). These are called "generalizing-promoting activities." The final study employed APOS Theory and used a Genetic Decomposition as the foundation of the instruction for a lesson on two-variable functions (Kabael, 2011). The instruction was mostly in lecture format but also utilized several tasks that required perceiving functional situations in an algebraic or graphic representation.

Indicators of Success

The review of the literature revealed that not all studies had a clear definition of what was deemed a successful teaching of generalization. Often, results reported and mentioned that some participants showed signs of generalization without a description of how that success manifested itself. Five generalizing actions mentioned by Ellis (2007a) included identifying properties of mathematical objects, extending one's reasoning to a larger range of cases from where it originated, searching for a repeated performance in a procedure, deriving a generalization from other known properties, and determining a rule as a verbalization or mathematical statement.

With the knowledge that generalization in the literature takes on numerous definitions, it is not surprising that authors see indicators of success differently. Furthermore, in some of the reviewed articles, even though generalization was being taught, the method, more so than the results, was more important to the writers. Three major sub-themes emerged when looking at the literature for indicators of success, with the remaining descriptions being grouped into a miscellaneous category. Those sub-themes include determining a formula, describing a pattern with words, and answering math facts. Following is a summary of those findings.

Determining a Formula

In fifteen of the reviewed articles, participants determined a rule while completing a task in which they expressed it as a mathematical equation or an algebraic expression. In four of these cases (Ellis, 2009; Hallinen et al., 2021; Lannin, 2006; Pinto; 2019), this seemed to be the goal of the study. Most of the other studies used patterning tasks with the explicit goal of pushing the students to generalize later iterations of the pattern. In several cases, more than one formula was suitable. As an example, in the W-Dot problem mentioned previously, some students developed an expression of $4n + 1$, and other students produced $4(n + 1) – 3$, representing different generalizations of the problem (Rivera & Becker, 2008). Patterns were almost always generalized with an algebraic expression. In one case from a third-grade classroom, a different goal was achieved when students simply used a variable to represent a quantity (Blanton et al., 2016).

Describing a Pattern with Words

Describing the end behavior of a pattern verbally was an indicator of success in seven articles. Only in two of those (Aparicio Landa et al., 2021; Yao & Manouchehri, 2019) were the learners above the fifth-grade level. Since most primary school learners have not been taught algebra, then it is reasonable that generalization success might be a verbal description of pattern behavior. As an example, in the toilet paper scenario previously mentioned, third graders named the number (26) located at a position (13) by saying it was "times two" (Ferrara & Sinclair, 2016,

p. 12). Another student later generalized the entire process by saying, “You try to, you do the double” (p.12).

Answering Math Facts

Answering math facts appeared to be the premise of success in each of the articles where the teaching method was by drills. In Codding et al. (2010), although never defined, generalization takes on meaning that the student can answer multiplication problems correctly after much drill and practice. As a reminder, this study employed the incremental rehearsal method where students would drill math facts, then practice those they had missed, and then drill some more. Naturally, then, success would be defined by answering those math facts correctly. They summarize: “Drill may be an important step towards building fluency but composite practice is likely needed to yield response maintenance and generalization” (Codding et al, 2010, p. 103).

Miscellaneous

Other measures of success range from the ability to perform mental math calculations (Hill et al., 2015; Laski et al., 2021) to making conjectures and writing proofs (Fahlgren & Brunström, 2014). One study used a pre- and post-test called the Pattern and Structure Assessment (PASA) to give a numerical score and then translated that to a level of structural development as described in PASMAP (Mulligan et al., 2020). Success was noted when half of the students in the study moved from the pre-structural level to the emergent level.

Conclusion

This review showed the most common method of teaching generalization is through the use of patterning tasks. More commonly used in the primary grades, these tasks encourage students to observe a pattern, extend the pattern, analyze the nature of the relationship, make predictions for larger stages, and then create a rule that will generalize the nature of the growing pattern (Hourigan & Leavy, 2015). Generalization was also shown to be taught with direct feedback, by using manipulatives, and through the use of technology (like DGE’s). Instructors very often guide students through tasks with reflective conversations as well, moving students to think about the processes they are using when making decisions about how to advance through a task. It was interesting to note, in the screening process, that many articles written about generalization are making an evaluation of generalization occurring in a classroom after the fact, but more in a “caught, not taught” scenario.

The primary indicator of a successful generalization was shown to be an algebraic formula or expression that can be used to name an extended term in a pattern. However, formulas are not necessary for generalizations to occur. Certain studies (Hill et al., 2015; Laski et al., 2021; Rich et al., 2107) advocated that being able to solve problems after targeted instruction about that type of problem is evidence of generalization. It is clear that the ideas about what constitutes successful generalization are many. Certainly, lessons which drive learners to use general expressions about topics they are studying are fundamental if generalization is to be successful.

In that generalization is so extremely important in the learning of mathematics, an emphasis must remain on teaching it in the classroom. Very few explicit methods of teaching generalization are in place or are ever even attempted. Given that this certainly needs to change, researchers should continue to look for methods that will succeed in teaching this important process.

References

References marked with an (*) were articles included in the systematic review.

*Aparicio Landa, E., Sosa Moguel, L., & Cabañas-Sánchez, G. (2021). Reflective conversation and knowledge development in pre-service teachers: The case of mathematical generalization. *International Journal of Education in Mathematics, Science and Technology, 9*(1), 40-62.

*Apsari, R., Putri, R., Abels, M., & Prayitno, S. (2020). Geometry representation to develop algebraic thinking: A recommendation for a pattern investigation in pre-algebra class. *Journal on Mathematics Education, 11*(1), 45-58.

*Blanton, M., Gardiner, A., Isler, I., Stephens, A., & Knuth, E. (2016). A longitudinal study of elementary students' use of variable notation to represent mathematical generalizations [Paper presentation]. The Annual Meeting of the American Educational Research Association, Washington, DC. https://www.aera.net/ Publications/Online-Paper-Repository/AERA-Online-Paper-Repository

Borji, V., & Voskoglou, M. (2016). Applying the APOS theory to study the student understanding of the polar coordinates. *American Journal of Educational Research*, *4*(16), 1149-1156. doi: 10.12691/education-4-16-5

Braun, V. & Clarke, V. (2012). Thematic analysis. In H. Cooper (Ed.), *The handbook of research methods in psychology* (pp. 57-71). American Psychological Association.

Carpenter, T. P., & Franke, M. L. (2001). Developing algebraic reasoning in the elementary school: Generalization and proof. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference: The future of the teaching and learning of algebra* (pp. 155-162). Melbourne, Australia: The University of Melbourne.

*Codding, R., Archer, J., & Connell, J. (2010). A systematic replication and extension of using incremental rehearsal to improve multiplication skills: An investigation of generalization. *Journal of Behavioral Education*, *19*, 93-105. https://doi.org/10.1007/s10864-010-9102-9

Davydov, V. V. (1990). *Soviet studies in mathematics education* (Vol. 2; J. Kilpatrick, Ed., J. Teller, Trans.). National Council of Teachers of Mathematics. (Original work published 1972).

Dörfler, W. (1991). Forms and means of generalization in mathematics. In A. J. Bishop (Ed.), *Mathematical knowledge: Its growth through teaching* (pp. 63–85). Kluwer. doi:10.1007/978-94-017-2195-0_4

Ellis, A. B. (2011). Generalizing-promoting actions: How classroom collaborations can support students' mathematical generalizations. *Journal for Research in Mathematics Education, 42*(4), 308-345. https://doi.org/10.5951/jresematheduc.42.4.0308

*Ellis, A.B. (2007a). The influence of reasoning with emergent quantities on students' generalizations. *Cognition and Instruction*, *25*(4), 439-478. https://doi.org/10.1080/ 07370000701632397

*Ellis, A.B. (2009). Patterns, quantities, and linear functions. *Mathematics Teaching in the Middle School*, *14*(8), 482-491.

Ellis, A.B. (2007b). A taxonomy for categorizing generalizations: Generalizing actions and reflection generalizations. *The Journal of the Learning Sciences*, *16*(2), 221-262. https://doi.org/10.1080/10508400701193705

Ellis, A.B., Tillema, E., Lockwood, E., & Moore, K. (2017). Generalization across domains: The relating-forming-extending generalization framework. In E. Galindo and J. Newton (Eds.), *Proceedings of the 39th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 677-684). Indianapolis, IN: Hoosier Association of Mathematics Teacher Educators.

*Fahlgren, M., & Brunström, M. (2014). A model for task design with focus on exploration, explanation, and generalization in a dynamic geometry environment. *Technology, Knowledge and Learning*, *19*, 287-315. https://doi.org/10.1007/s10758-014-9213-9

*Ferrara, F., & Sinclair, N. (2016). An early algebra approach to pattern generalisation: Actualising the virtual through words, gestures and toilet paper. *Educational Studies in Mathematics*, *92*, 1-19.

Green, S., Higgins J. P .T., Alderson, P., Clarke, M., Mulrow, C.D., & Oxman, A.D. (2008). Introduction. In J. P .T. Higgins and S. Green (Eds.), *Cochrane handbook for systematic reviews of interventions* (pp. 3-9). John Wiley & Sons.

*Hallinen, N., Sprague, L., Blair, K., Adler, R., & Newcombe, N. (2021). Finding formulas: Does active search facilitate appropriate generalization?. *Cognitive Research: Principles and Implications*, *6*, 1-18. https://doi.org/10.1186/s41235-021-00316-y

Hannes, K., Claes, L., & The Belgian Campbell Group. (2007). Learn to read and write systematic reviews: The Belgian Campbell Group. *Research on Social Work Practice, 17*(6), 748-753.

Harel, G., & Tall, D. 0. (1991). The general, the abstract, and the generic in advanced mathematics. *For the Learning of Mathematics, 11*(1), 38-42.

*Hill, T., Lannin, J., & van Garderen, D. (2015). Promoting and assessing mathematical generalising. *Australian Primary Mathematics Classroom*, *20*(4), 3-8. https://files. eric.ed.gov/fulltext/EJ1093246.pdf

*Hourigan, M., & Leavy, A. (2015). Geometric growing patterns: What's the rule?. *Australian Primary Mathematics Classroom*, *20*(4), 31-39. https://files.eric.ed.gov/fulltext/ EJ1093258.pdf

*Kabael, T. (2011). Generalizing single variable functions to two-variable functions, function machine and APOS. *Educational Sciences: Theory and Practice*, *11*(1), 484-499. https://files.eric.ed.gov/fulltext/EJ919912.pdf

*Lannin, J., Barker, D., & Townsend, B. (2006). Recursive and explicit rules: How can we build student algebraic understanding?. *The Journal of Mathematical Behavior*, *25*(4), 299-317. https://doi.org/10.1016/j.jmathb.2006.11.004

*Laski, E., Ermakova, A., Vasilyeva, M., & Halloran, K. (2023). Effects of using one or more manipulatives on strategy mastery and generalization. *The Journal of Experimental Education*, *91*(2), 230-248. https://doi.org/10.1080/00220973.2021.1973358

*Martino, A.M., & Maher, C.A. (1999). Teacher questioning to promote justification and generalization in mathematics: What research practice has taught us. *Journal of Mathematical Behavior*, *18*, 53-78. https://doi.org/10.1016/S0732-3123(99)00017-6

Mason, J. (1996). Expressing generality and roots of algebra. In N. Bernarz, C. Kiernan, and L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 65-86). Kluwer.

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically*. Pearson.

*Mata-Pereira, J., & da Ponte, J. (2017). Enhancing students' mathematical reasoning in the classroom: Teacher actions facilitating generalization and justification. *Educational Studies in Mathematics, 96*(2), 169-186. https://doi.org/10.1007/s10649-017-9773-4

*Mavrikis, M., Noss, R., Hoyles, C., & Geraniou, E. (2013). Sowing the seeds of algebraic generalization: Designing epistemic affordances for an intelligent microworld. *Journal of Computer Assisted Learning*, *29*(1), 68-84. https://doi.org/10.1111/j.1365-2729.2011.00469.x

*Miller, J. (2019). STEM education in the primary years to support mathematical thinking: Using coding to identify mathematical structures and patterns. *ZDM*, *51*(6), 915-927. https://doi.org/10.1007/s11858-019-01096-y

*Miller, K., Skinner, C., Gibby, L., Galyon, C., & Meadows-Allen, S. (2011). Evaluating generalization of addition-fact fluency using the taped-problems procedure in a second-grade classroom. *Journal of Behavioral Education*, *20*, 203-220. https://psycnet.apa. org/doi/10.1007/s10864-011-9126-9

Mitchelmore, M. C. (2002). *The role of abstraction and generalization in the development of mathematical knowledge* [Paper presentation]. The East Asia Regional Conference on Mathematics Education (EARCOME) (2nd) and the Southeast Asian Conference on Mathematics Education (SEACME). Singapore. https://files.eric.ed.gov/ fulltext/ ED466962.pdf

Moher, D., Shamseer, L., Clarke, M., Ghersi, D., Liberati, A., Petticrew, M., Shekelle, P., Stewart, L., & PRISMA-P Group. (2015) Preferred reporting items for systematic review and meta-analysis protocols (PRISMA-P) 2015 statement. *Systematic Reviews, 4*, 1-9.

*Mulligan, J., Mitchelmore, M., Kemp, C., Marston, J., & Highfield, K. (2008). Encouraging mathematical thinking through pattern and structure: An intervention in the first year of schooling. *Australian Primary Mathematics Classroom*, *13*(3), 10-15.

*Mulligan, J., Oslington, G., & English, L. (2020). Supporting early mathematical development through a 'pattern and structure' intervention program. *ZDM, 52*, 663-676. doi:10.1007/978-94-007-6440-8_3

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* NCTM.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. NCTM.

*Pauletti, K., & Zaslavsky, O. (2018). Spontaneous generalizations through example-based reasoning in a collaborative setting. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.). (2018). *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 572-579). Greenville, SC: University of South Carolina & Clemson University.

*Pinto, E., & Cañadas, M. C. (2021). Generalizations of third and fifth graders within a functional approach to early algebra. *Mathematics Education Research Journal*, *33*(1), 113-134. https://doi.org/10.1007/s13394-019-00300-2

*Poncy, B. C., Duhon, G. J., Lee, S. B., & Key, A. (2010). Evaluation of techniques to promote generalization with basic math fact skills. *Journal of Behavioral Education*, *19*, 76-92. https://doi.org/10.1007/s10864-016-9262-3

*Rich, S., Duhon, G., & Reynolds, J. (2017). Improving the generalization of computer-based math fluency building through the use of sufficient stimulus exemplars. *Journal of Behavioral Education*, *26*, 123-136. https://doi.org/10.1007/s10864-016-9262-3

*Richardson, K., Berenson, S., & Staley, K. (2009). Prospective elementary teachers use of representation to reason algebraically. *The Journal of Mathematical Behavior*, *28*(2-3), 188-199. https://doi.org/10.1016/j.jmathb.2009.09.002

*Rivera, F. D. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics*, *73*(3), 297-328. https://doi.org/10.1007/s10649-009-9222-0

*Rivera, F. D., & Becker, J. R. (2008). Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns. *ZDM, 40*, 65-82. https://doi.org/10.1007/s11858-007-0062-z

*Rivera, F. D., & Becker, J. R. (2016). Middle school students' patterning performance on semi-free generalization tasks. *The Journal of Mathematical Behavior, 43*, 53-69. https://doi.org/10.1016/j.jmathb.2016.05.002

*Sosa Moguel, L., Aparicio Landa, E., & Cabañas-Sanchez, M. (2019). Characterization of inductive reasoning in middle school mathematics teachers in a generalization task. *International Electronic Journal of Mathematics Education, 14*(3), 563-581. https://doi.org/10.29333/iejme/5769

*Sriraman, B., & Adrian, H. (2004). The pedagogical value and the interdisciplinary nature of inductive processes in forming generalizations: Reflections from the classroom. *Interchange*, *35*, 407-422. https://doi.org/10.1007/BF02698891

Strech, D., & Sofaer, N. (2012). How to write a systematic review of reasons. *Journal of Medical Ethics, 38*, 121-126. doi:10.1136/medethics-2011-100096

*Stephens, A., Fonger, N. L., Blanton, M., & Knuth, E. (2016). Elementary students' generalization and representation of functional relationships: A learning progressions approach (Paper presentation). The Annual Meeting of the American Educational Research Association. Washington, D. C. https://files.eric.ed.gov/fulltext/ED566523.pdf

*Stephens, A., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning*, *19*(3), 143-166. https://doi.org/10.1080/ 10986065.2017.1328636

*Strachota, S. (2020). Generalizing in the context of an early algebra intervention (La generalización en el contexto de una intervención algebraica temprana). *Journal for the Study of Education and Development*, *43*(2), 347-394. https://doi.org/10.1080/ 02103702.2020.1732611

*Widjaja, W., Vale, C., Herbert, S., Loong, E., & Bragg, L. (2021). Linking comparing and contrasting, generalising and justifying: A case study of primary students' levels of justifying. *Mathematics Education Research Journal*, *33*, 321-343. https://doi.org/ 10.1007/s13394-019-00306-w

Wright, R., Brand, R., Dunn, W., & Spindler, K. (2007). How to write a systematic review. *Clinical Orthopaedics and Related Research*, *455*, 23-39. doi: 10.1097/BLO .0b013e31802c9098

*Yao, X. (2022). Representation transformations in transition from empirical to structural generalization. *The Journal of Mathematical Behavior*, *66*, 100964. https://doi.org/ 10.1016/j.jmathb.2022.100964

*Yao, X., & Manouchehri, A. (2019). Middle school students' generalizations about properties of geometric transformations in a dynamic geometry environment. *The Journal of Mathematical Behavior*, *55*, 100703. https://doi.org/10.1016/j.jmathb.2019.04.002

Author Information

Jay L. Jackson	Cynthia L. Stenger
https://orcid.org/0009-0000-6611-7947	https://orcid.org/0000-0001-7014-4243
University of North Alabama	University of North Alabama
United States	United States
Contact e-mail: *jljackson3@una.edu*	