Collective Mathematical Progress in an Introductory Calculus Course during the Treatment of the Quadratic Function

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Article Info	Abstract
	This study characterizes the classroom mathematical practices that support the collective mathematical progress during the treatment of a quadratic function in an Introductory Calculus course comprised of 15 students from eleventh grade (from 16 to 18 years old). A teaching experiment that discussed two variational situations was designed for the treatment of the quadratic function. This paper only reports data of one these situations. The analysis of the empirical data involved an analytical approach that used Toulmin's model of argumentation and two criteria to inform the classroom mathematical practices. Results correspond with two mathematical practices that supported a collective mathematical progress: 1) analyze and quantify what changes in a variational situation; and 2) generate and interpret representations of how the variable magnitudes change in a variational situation. Three connected elements were identified in the evolution of the mathematical practices: time, structure and mathematics. These elements enabled the establishment of a model of collective mathematical progress based on empirical evidence; this model documents a path students could follow to learn quadratic function. The proposed model has the quality of being explicative and the ability to adjust to interact with other students.

Introduction

The mathematics education community has widely considered the interactions inside the classroom as an integral part of learning in a sociocultural sense (Voigt, 1994; Bowers, Cobb & McClain, 1999; Stephan & Rasmussen, 2002; Güven & Dede, 2017). These sociocultural interactions are mediators of psychological processes, such as the mathematical reasoning that takes place between students. It is assumed that the observable *mathematical progress* in class is a path to analyze the mathematical learning from an individual or collective activity. This progress is a construct characterized in the sociological framework of Cobb and Yackel (1996) and their associated works (e.g. Bowers et al., 1999; Cobb, Stephan, McClain & Gravemeijer, 2001; Stephan & Rasmussen, 2002; Rasmussen, Wawro & Zandieh, 2015); it is understood as an evolution of the mathematical reasoning used to solve mathematics tasks (Rasmussen et al., 2015).

The studies of the collective mathematical progress in reference to the classroom mathematical practices are based on the hypothesis that it is possible to document learning trajectories from instruction (Cobb, 1999; Cobb et al., 2001) focusing on the social productions of ways of reasoning, argumentation and using tools that will eventually be institutionalized and placed beyond justifications (Bowers et al., 1999; Cobb, 1999). Thus, the evolution of the mathematical practices has been analyzed and reported in different classroom contexts and educational levels; for example, elementary school (Cobb, Yackel & Wood, 1992; McClain, Cobb, Gravemeijer & Estes, 1999; Bowers et al., 1999; Cobb et al., 2001), middle school (Cobb, 1999) and higher education (Stephan & Rasmussen, 2002; Rasmussen, Stephan & Allen, 2004; Rasmussen, Zandieh & Wawro, 2009; Rasmussen et al., 2015). However, this research is limited in high school.

As a consequence, this work focuses on characterizing the classroom mathematical practices that support a collective mathematical progress in the treatment of a quadratic function in an Introductory Calculus course. This course is part of the high school curriculum.

Theoretical Background

The theoretical approach considered in this investigation is the sociocultural perspective of Cobb and Yackel (1996) and its interpretative framework of social and individual aspects that norm the interactions in the classroom. These aspects somehow determine the thoughts and actions of the individuals during the interaction and, as a consequence, the mathematical process that is taking place. This approach considers that the mathematical practices emerge from ideas and mathematical norms socially shared in class and expand the mathematical activity of the students to a more advanced level of reasoning.

Rasmussen et al. (2015) used the construct classroom mathematical practices as an extension of the interpretative framework of Cobb and Yackel (1996) to conceptualize the individual and collective mathematical progress of the students. These authors define *classroom mathematical practices* as:

> "Normative ways of reasoning that emerge as learners solve problems, explain their thinking, represent their ideas, and so on. By normative we mean […] that an idea or way of reasoning functions as if it is a mathematical truth in the classroom. This means that particular ideas or ways of reasoning are functioning in classroom discourse as if everyone has similar understandings, even though individual differences in understanding may exist" (Rasmussen et al., 2015, p. 262).

Mathematical practices are recognized and established through the interactions in the classroom. A specific interaction is argumentation (Krummheuer, 1995); it is understood as "interactions in the observed classroom that have to do with the intentional explication of the reasoning of a solution or after it" (p.231). To document the mathematical practices it is methodologically necessary to establish argumentations as a norm in the classroom (Cobb & Yackel, 1996) and the use of analytical approaches (e.g. Stephan & Rasmussen, 2002; Rasmussen & Stephan, 2008) that allow modeling of the explanations given in the collective activity of the students. The production of these mathematical practices in the classroom constitutes the collective, emergent and potentially idiosyncratic mathematical progress in a local community (Rasmussen et al., 2015). In this sense, examining the collective mathematical progress is studying the mathematical learning of a classroom community.

Methodology

A teaching experiment was conducted to study the nature of the classroom mathematical progress (Cobb, 2000), with the learning goal of understanding the concept of quadratic function under the proposal of Dolores (2013), focusing on the study of variation and change to treat the ideas and basic concepts of calculus, such as function. This author establishes that studying *which magnitudes change, how much and how they change* should be the central teaching and learning activity in variational situations; in this sense the concept of function is a tool to describe and quantify the change.

The teaching experiment was carried out in six sessions, each of approximately 40 minutes in an Introductory Calculus course. Two variational situations were discussed. This paper reports data of the first four sessions where the first situation was discussed.

Participants and Context of the Study

The experiment was carried out in a group of fifteen students of eleventh grade (between 16 and 18 years old), eleven men and four women, in a Colegio de Bachilleres (equivalent to high school grades 10-12) in Guerrero, México. The participants have already finished three mathematical courses: the first was Arithmetic and Algebra, the second was Geometry, Trigonometry and Probability and Statistics, and the third was Analytic Geometry. The students participated in this teaching experiment at the request of the teacher in charge of the course who was informed of the goal of this research.

The teaching experiment was guided by the main author of this document in the role of teacher. His role was to promote the development of mathematics and establish a legitimate argumentation as a norm in the classroom (Cobb & Yackel, 1996), this means that the intention was to promote in the students:

1) Explanations and justifications for their reasoning, make sense of the explanations of their classmates, and indicate agreement or disagreement, and

2) Share and judge what was considered as a different mathematical solution, a fancy and efficient mathematical solution, and an acceptable mathematical explanation and justification.

In addition, the teacher promoted the interaction in small groups and with the whole class respecting ideas and establishing points to focus the students in the situations. There were two moments in the classroom activity. The first one involved solving a variational situation in teams, and the second one the discussion of the solutions with the whole group. For this purpose, the participants were organized in teams and were comprised of three students per team: Team 1 by Leo, Fabián and Alba; Team 2 by Mauro, Omar and Pedro; Team 3 by Sara, Hugo and Juan; Team 4 by Teo, Gael and Eva; and Team 5 by Iris, Diana and Raúl. The names used in this paper are pseudonyms in order to respect their anonymity. These teams remained the same during the whole experiment.

Data Collection Instruments

Variational situation: Squares and areas
A square ABCD has a side length of 26 cm. The vertices of the inscribed square EFGH are at a distance of 2 cm from the vertices of the original square. Answer the following questions, based on this data.
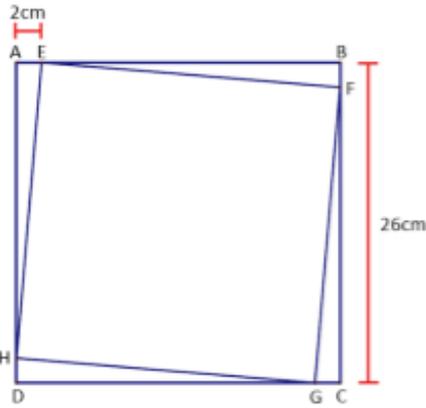
1. What is the area of EFGH when the distance from A to E is 2 cm?
2. Determine some squares inscribed in the square ABCD with smaller area than the one in the first question. How long are the distances from A to E to find them? What is the area of those squares?
3. Determine some squares inscribed in square ABCD with bigger area than the one in the first question. How long are the distances from A to E to find them? What is the area of those squares?
4. Determine some squares inscribed in square ABCD with equal area to the one in the first question. How long are the distances from A to E to find them? What is the area of those squares?
5. Determine the area of EFGH when the distance from A to E is:
   a) 10cm
   b) 16cm
   c) 13cm
6. Determine some squares inscribed in square ABCD with areas equal to the one in question 5 subsection c). How long are the distances from A to E to find them? What is the area of those squares?
7. Determine some squares inscribed in square ABCD with equal areas. How long are the distances from A to E to find them? What is the area of those squares?
8. What is the least possible area of a square inscribed in square ABCD? How long is the distance from A to E?
9. Describe how the area of the inscribed square changes while the AE distance is modified.
10. How much does the area of the inscribed square change for each centimeter of the AE distance?
11. How is the area of the square EFGH determined knowing the AE distance?

Figure 1. Variational situation

The design of the variational situation (Figure 1) was based on some principles of the instructional theory of Realistic Mathematics Education (Van den Heuvel-Panhuizen & Drijvers, 2014):

1) Of reality. It is based on a significant problematic situation for the students, a variational situation (adapted from the work of Illuzi & Sessa, 2014, p. 39) in a geometric context associated with geometrical figures and measures of its area.

2) Of activity. Students' activity involved the mathematization (or organization) of the variational situation by using the function as a tool to describe the changes of the area of a square inscribed in another square while modifying the distances between the vertices of both squares. This activity involves the use of the quadratic function in the form $y = ax^2 + bx + c$ with $a > 0$ and the transition between its tabular, algebraic and graphic representations.

3) Of level. The mathematization of the variational situation required the students to get through different levels of comprehension, from intuitive solutions connected with the context of the situation (geometrical) to acquiring information about the relationships between concepts and strategies.

The data considered in this investigation were collected from the written productions of the students, the images captured from the blackboard and the transcriptions from the video records of the group discussions of all sessions in class.

Data Analysis

The mathematical practices were identified through the argumentation produced in the collective activity of the students. The analytical approach of Stephan and Rasmussen (2002) was considered for the analysis of the empirical data to identify and characterize the mathematical practices that supported a collective mathematical progress. In this approach, Toulmin's argumentative model is used to analyze the structure and ideas expressed in the argumentations of the students and to monitor how these ideas work in the discourse of the students with the aim to identify the ideas that become part of the normative reasoning in the classroom.

There are three phases in the analysis of the empirical data. The first phase was the reconstruction of the group discussions in all sessions through Toulmin's argumentative model (Figure 2), whose basic structure consists of data, claim and warrant; it also includes backing, qualifiers and rebuttals. As a consequence, a set of coded arguments is obtained. The collection of all coded arguments results in an argumentation log of the four sessions.
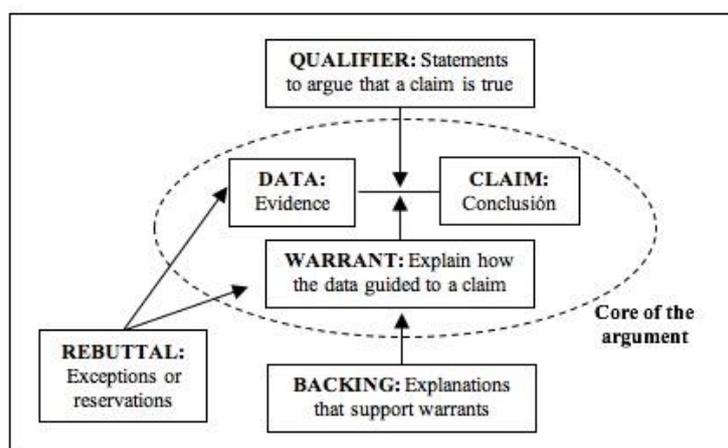
Figure 2. Toulmin's argumentative model. Adapted from Rasmussen et al. (2015, p.263)

The second phase examines all sessions in class based on the records of argumentation as data in order to identify the mathematical ideas expressed in the arguments that become the normative ways of reasoning in class. The criteria used to determine if a way of reasoning transforms into normative are (Stephan & Rasmussen, 2002, p.462):

Criterion 1 (C1): the backings and/or warrants for an argumentation no longer appear in students' explanations and therefore the mathematical idea expressed in the core of the argument stands as self-evident, or

Criterion 2 (C2): any of the four parts of an argument (data, warrant, claim, backing) shift position (i.e., function) within subsequent arguments and are unchallenged.

The third and last phase analyzed the normative ways of reasoning identified in the previous phase and organized them around the common mathematical activities carried out by the students. These mathematical activities are the ones recognized as classroom mathematical activities.

Collective Mathematical Progress

Six normative ways of reasoning in the collective activities of students in the mathematization of a variational situation that involved the quadratic function were identified based on the results of the teaching experiment in the classroom. The emergence of these normative ways of reasoning gives evidence of the constitution of classroom mathematical practices, and therefore, reveals the mathematical progress of the students. Two specific mathematical practices that support the collective mathematical progress in the treatment of the quadratic function are detected in this research.

Normative Ways of Reasoning

Table 1 shows the six Normative Ways of Reasoning (NWR) that emerged in the collective activity of the students and the criteria satisfied.

Table 1. Normative ways of reasoning identified in the study

Normative ways of reasoning	Criterion
NWR1: The square of the length of the hypotenuse equals the area of the generated inscribed square.	C1
NWR2: The length of $\overline{AE}$ changes from 0 to 26cm.	C2
NWR3: Measurements of equivalent areas.	C1
NWR4: The algebraic expressions that explain the measurements of the area of the inscribed square in terms of the measure $\overline{AE}$ are based on the Pythagoras theorem (PT) and the Differences in the Area (DA).	C1
NWR5: First, the area of the squares decreases, reaches the minimum area and then increases.	C2
NWR6: A parabola is the graphical representation of the behavior of the changes of the inscribed squares generated by modifying the length $\overline{AE}$.	C2

The next sections document the six normative ways of reasoning.

NWR1: The square of the length of the hypotenuse equals the area of the generated inscribed square

The students presented to and discussed with the group the methods they used to determine the areas of the inscribed squares during session two (the teacher promoted this discussion). As a result, it is recognized that the students supported their arguments with geometrical representations of particular cases of the situation (e.g. Figure 3). In this context, they identified that the sides (ℓ) of the inscribed square match with the hypotenuse (c) of the four right triangles formed in the squares ABCD and EFGH, this means that $\ell = c$; so this magnitude could be determined with the PT. After the length of the side was known, they used the basic formula $\ell \times \ell$ to determine the area of the square.

Sara described the reasoning followed by her team to determine the areas of specific inscribed squares using a geometrical representation of the situation on the blackboard (Figure 3). They used the PT and the formula $\ell \times \ell$. Argument 1 (Figure 4) rebuilds the last step of this reasoning.

In the same stage of classroom debate, the students also recognized that $c^2 = \ell^2 = A$; Fabián justified this mathematical relation after the teacher's questioning.

[96]	Teacher:	You say that the value of the square of the hypotenuse equals the area of the inscribed square… why?
[97]	Fabián:	Because… the square of the hypotenuse [points out the expression $c^2 = 338$ on the blackboard, Figure 3]... and the hypotenuse comes to be the inscribed square… and the area is obtained multiplying one side by the other side.
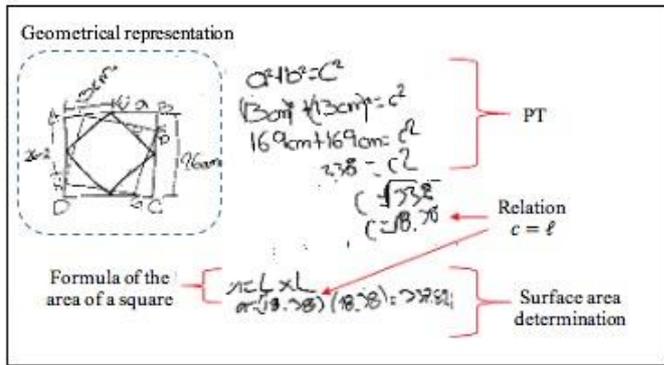
Figure 3. Procedure method shared by Sara   Figure 4. Argument 1. Use of $A = \ell \times \ell$
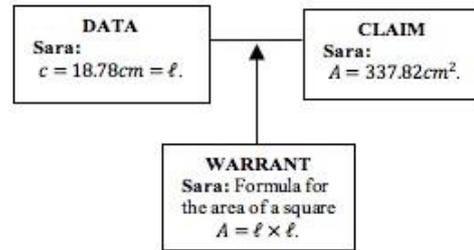
From now on, this mathematical relation was used as a warrant in the determination of the areas of inscribed squares to determine which one was smaller, bigger or equal; therefore, it is considered as a shared mathematical relation applied after using the PT to determine the value of *c*. The reasoning of Eva's team (Figure 5) is the empirical evidence of the evolution of the arguments used to justify their ways to determine the areas of specific inscribed squares. First, they used $\ell \times \ell$ and then $c^2 = \ell^2 = A$. This empirical evidence shows that C1 is satisfied and therefore the NWR that was constituted is: *The square of the length of the hypotenuse equals the area of the generated inscribed squared.*

The argument 2 (Figure 6) rebuilds the final part of the reasoning shared by Eva.
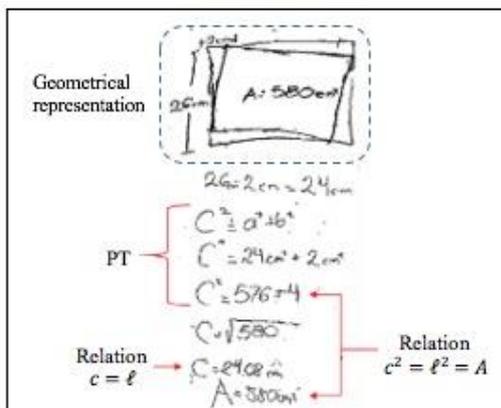
Figure 5. Use of PT, $c = \ell$ and $c^2 = \ell^2 = A$ Figure 6. Argument 2. Use of $c^2 = \ell^2 = A$
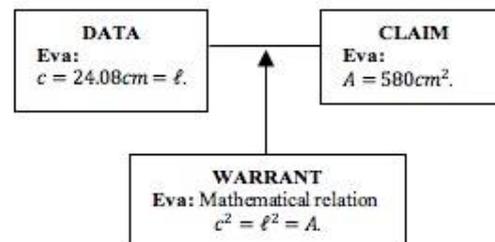
It is recognized that the students identified the two variables of the situation (the length $\overline{AE}$ and the area of the square inscribed in the square ABCD) and their dependence relation in this process of consolidation of NWR1. They also differentiate between implicit, known and unknown data and use this differentiation to establish procedures and mathematical relations, like the PT, $\ell = c$ and $c^2 = \ell^2 = A$. These relations allowed them to determine the areas of the inscribed squares generated by certain measures of the segment AE.

NWR2: The length of $\overline{AE}$ changes from 0 to $26cm$

Session two urged the students to determine a set of measurements for segment AE to generate inscribed squares with a bigger or smaller area than the one in the inscribed area generated when $\overline{AE} = 2cm$ ($580cm^2$). From the analysis of this stage (excerpts 138-148) and the geometrical representations of particular cases of the situation, it is identified that the students recognized that the length of $\overline{AE}$ could take different values (or change) from 0 to 26 cm. and generate the squares EFGH inscribed in the square ABCD. Initially, the students considered that these values correspond with positive integer numbers.

[138]   Teacher: (…) What would be the range of the lengths of AE to generate squares inscribed in ABCD? What is the size of the increments when AE changes?
[139]   Fabián: From 1 to 26.

[140]	Alba:	Couldn't it be from 0 to 26?
[141]	Teacher:	If AE were 0, what would the area of the inscribed square be?
[142]	Fabián:	676.
[143]	Teacher:	Why?
[144]	Fabián:	Because it is generating the same square ABCD.
[145]	Teacher:	And if the length of AE is 26?
[146]	Mauro:	It would be the same… the same square [he refers to the square ABCD].
[147]	Teacher:	The same! 0 and 26 restricts the values of AE?
[148]	Fabián:	Yes… AE changes from 0 to 26 centimeters.

In session four, the teacher promoted a new discussion on the values that the segment AE could take by challenging the students to consider if negative and decimal numbers could be used. The students recognized they could also use the decimal numbers because these values were "inside the measures that the segment AE could take", this warrant was exposed by Mauro. This means that the segment AE could take values in the closed interval from 0 to 26, the students understood the limit points of this interval as the cases where the square EFGH is not inscribed in the square ABCD anymore and matches the square ABCD.

The claim (excerpt 148) was used as warrant in other arguments like the one expressed by Mauro and argument 3 (Figure 7). This empirical evidence shows that C2 is satisfied and therefore a second NWR that was constituted was: *The length of AE changes from 0 to 26 cm*.
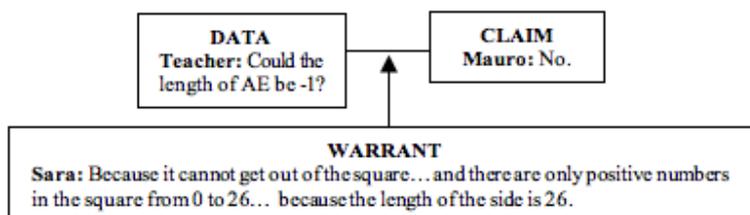
Figure 7. Argument 3. The length of $\overline{AE}$ cannot be negative

NWR3: Measurements of equivalent areas

The new challenge in session three (excerpts 122-133) was to determine pairs of values of length of $\overline{AE}$ that could generate inscribed squares of the same area. In this context, the students supported their solutions with geometrical representations of particular cases of the situation and recognized:
1) The symmetry of point E with respect to the middle point of the side AB that defines the other length ($\overline{AE'}$) that $\overline{AE}$ could take. The students expressed this symmetric relation in terms of "the values are inverted".
2) A rule that applies to these pairs of lengths ($\overline{AE}$ and $\overline{AE'}$), it was expressed in terms of "the sum of these pairs should be 26cm".

[122]	Teacher:	The length of AE is one centimeter and the area is 626. Is there any other measurement of AE that has the same area of 626?
[123]	Mauro:	Do you mean that AE has a different value with the same area of 626?
[124]	Fabián:	The values are inverted.
[125]	Teacher:	The values are inverted? How long would AE be?
[126]	Fabián, Omar:	25.
[127]	Teacher:	Are they always 25 and one?
[128]	Fabián:	Yes… that is why the values are inverted.
[129]	Teacher:	Are there other values?
[130]	Eva:	5cm and 21cm with equal area 466cm² [She writes it on the blackboard].
[131]	Teacher:	You show another pair of values for AE, what is happening with these pair of values? Is there any pattern?
[132]	Omar:	The sum of these pairs should be 26cm.

Right away, the teacher questioned the students about the square with minimum area that could be inscribed in the square ABCD. They identified that it would be the one generated with $\overline{AE} = 13cm$ (middle point of the segment AB) and area equals to $338cm^2$.

Argument 4 (Figure 8) describes the reasoning followed by Raúl's team to determine the square with minimum area. Their reasoning is supported in the comparison of the areas determined with each length of $\overline{AE}$ assigned. The tabular representation that expresses the correspondence between these two variables endorsed their reasoning to recognize: 1) the square of minimum area; and 2) the values of the areas repeat themselves from the minimum area, that is to say, they are equivalents. This is translated as the areas of the squares inscribed in the square ABCD could take values in the closed interval with limit points 338 and 626.
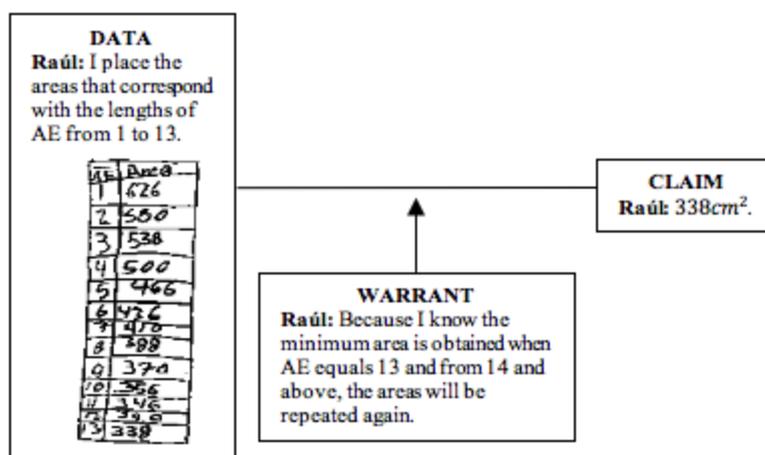
Figure 8. Argument 4. The minimum area in the squares inscribed in ABCD

This way of reasoning is empirical evidence of how the arguments to justify the minimum area of the squares inscribed in ABCD evolved. The symmetry of point E with respect to the middle point of the side AB and the rule that the sum of these pairs of $\overline{AE}$ should be 26 cm were used to support that the values of the length of $\overline{AE}$ "from 14 and above" the areas of the inscribed squares "will be repeated again". Therefore, C1 is satisfied and the emergence of a third NWR is constituted. This NWR refers to: *Measurements of equivalent areas.*

NWR4: The algebraic expressions that explain the measurements of the area of the inscribed square in terms of the measure $\overline{AE}$ are based on the PT and the DA.

The teacher challenged the students to determine an algebraic expression that explains the area of the inscribed squares generated each time the length of $\overline{AE}$ is modified during a classroom debate in session three. After a period of time, Mauro shared the reasoning followed by his team to determine two expressions. Two methods supported their reasoning: PT and DA.

Argument 5 (Figure 9) shows the reasoning followed by Mauro's team to establish two algebraic expressions that explain the rule of correspondence between variable magnitudes embedded in the situation. This reasoning results from: 1) recognizing that the length $\overline{AE}$ takes values between 0 and 26 cm.; 2) establishing that the area of the square inscribed in the square ABCD depends on the length of $\overline{AE}$; 3) representing the length of $\overline{AE}$ with the variable x and the area of the inscribed square as $A(x)$; and 4) substituting both variables, x and $A(x)$, in the relations PT or DA, considering the known values.

Four of the five teams used PT to form an algebraic expression of the relation of correspondence between the variable magnitudes embedded in the variational situation. The two expressions showed by Mauro were not considered as shared until the discussion in the fourth session when the teacher asked the students to write the algebraic expressions that explained the relationship between the variable magnitudes in the situation analyzed. Eva and Omar wrote the expressions based on PT and DA, respectively, but without any reference to the warrant showed in argument 5. This empirical evidence shows that C1 is satisfied and therefore the fourth NWR constituted is: *The algebraic expressions that explain the measurements of the area of the inscribed square in terms of the measure $\overline{AE}$ are based on the PT and the DA.*
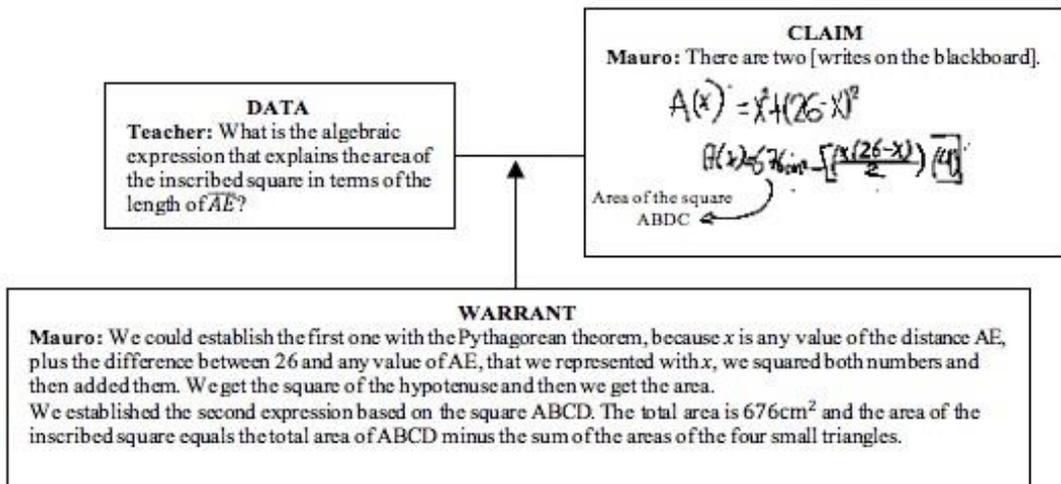
Figure 9. Argument 5. Algebraic expressions based on PT and DA

NWR5: First, the area of the square decreases, reaches the minimum area and then increases

The analysis of the observed regularity in the pairs of inscribed squares with the same area continued for a whole debate in session four. The students supported their arguments in a tabular representation (Figure 10) presented by Raúl, Fabián and Omar. They shared their reasoning followed to identify these equivalences (excerpts 212-234).

At another moment in the debate, the teacher focused the students on the analysis of the behavior of the changes in the area of the squares inscribed in ABCD. Fabián, Omar and Alba supported their arguments in the table to explain that for the first lengths of $\overline{AE}$, the area decreases and then, after the minimum area, the length increases. They used the following terms to refer to the behavior of the changes in the area: increase (or goes up or raises), decrease (or goes down or diminishes) and the minimum area. In short, the teacher wrote the intervals at which the areas increase and decrease in the tabular representation on the blackboard (Figure 10).

Figure 10. Tabular representation

[212]	Teacher:	You showed lengths of AE from one to thirteen centimeters [point out the table in Figure 8] Why? Is there a reason? What happens with the areas?
[213]	All:	They repeat.
[214]	Teacher:	They repeat?
[215]	Fabián:	Because thirteen is half, and they repeat from bottom up.
[216]	Teacher:	If you take the other values [refers to the values of AE from 14 to 26cm], what would the area of the generated squares be?
[217]	Fabián:	The same ones… After 338, it would 340 and so on, because they repeat.
[…]		
[232]	Teacher:	And when the length of AE is 15cm?
[233]	Fabián:	346… and then 356… 370… 388…
[234]	Omar:	And so on.

[Raúl, Fabián and Omar present a tabular representation expressing the correspondence between the areas of the squares inscribed in the square ABCD and the lengths of $\overline{AE}$ from 0 to 26cm, Figure 10].

The reasoning of the students is supported by: 1) the correspondence relation identified based on the tabular representation between the area of each inscribed square generated from the assigned value of $\overline{AE}$; and 2) the behavior of the changes of the area of those squares identified at specific intervals of the lengths of $\overline{AE}$.

Argument 6 (Figure 11) rebuilds the last part of the reasoning generated by Fabián, Omar and Alba around the behavior of the changes of the areas of the squares inscribed in ABCD when the lengths of $\overline{AE}$ are modified. In this argument, it can be appreciated that the teacher summarized this discussion in a claim supported by the evidence presented in the data. The data was also used as claims in previous arguments. This empirical evidence shows that C2 is satisfied and therefore a fifth NWR constituted is: *First, the area of the squares decreases, reaches the minimum area and then increases.*
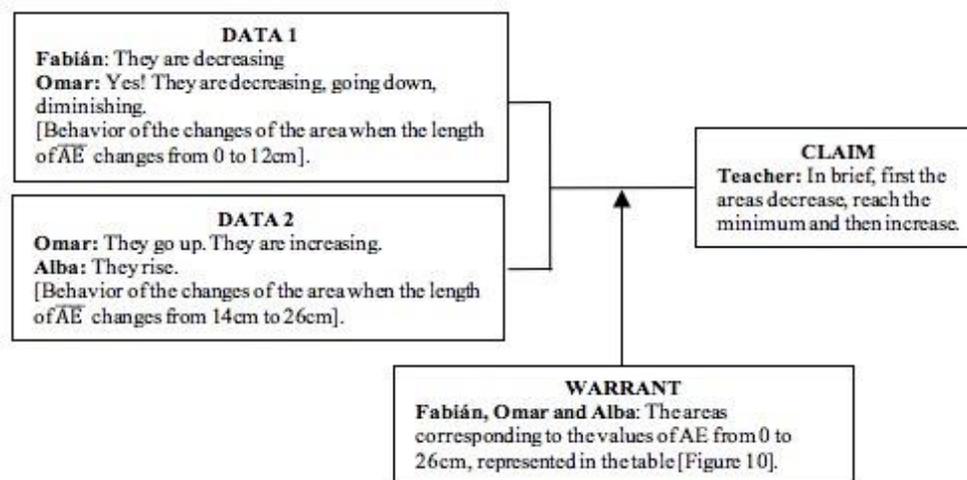
Figure 11. Argument 6. Behavior of the changes of the area

NWR6: A parabola is the graphical representation of the behavior of the changes of the inscribed squares generated by the length $\overline{AE}$

It was recognized how the areas of the squares inscribed in ABCD change when the lengths of $\overline{AE}$ are modified during the discussions of the students in one of the debates in session four. Fabián and Mauro associated this behavior with a parabola, describing it with hand movements like a U-shaped graph based on the table (Figure 10) and the shared NWR5. The teacher asked for a graphical representation on the blackboard that backs their gestural description.

Argument 7 (Figure 12) shows the reasoning followed by Fabián and Mauro in response to the challenge of the teacher.

At another time during the activity, the teacher urged the students to analyze the graphical representation of Omar and Fabian in the classroom. The classroom debate on the graph promoted the recognition of the following aspects:
1) Omar expressed that the behavior of the graph represented by a parabola "has a degree of 2";
2) Fabián made sure that the graphical representation matches the description of the behavior of the changes in the variables of the situation; and
3) Alba associated the vertex of the parabola with the "middle point" and the "minimum point" in the graph.

Argument 8 (Figure 13) describes Alba's reasoning. It can be acknowledged that the graphical representation acts as the data needed to establish a new claim; while in argument 7 (Figure 12) it acts as backing. This empirical evidence shows that C2 is satisfied and therefore a sixth NWR constituted is: *A parabola is the graphical representation of the behavior of the changes of the inscribed squares generated by modifying the length $\overline{AE}$.*
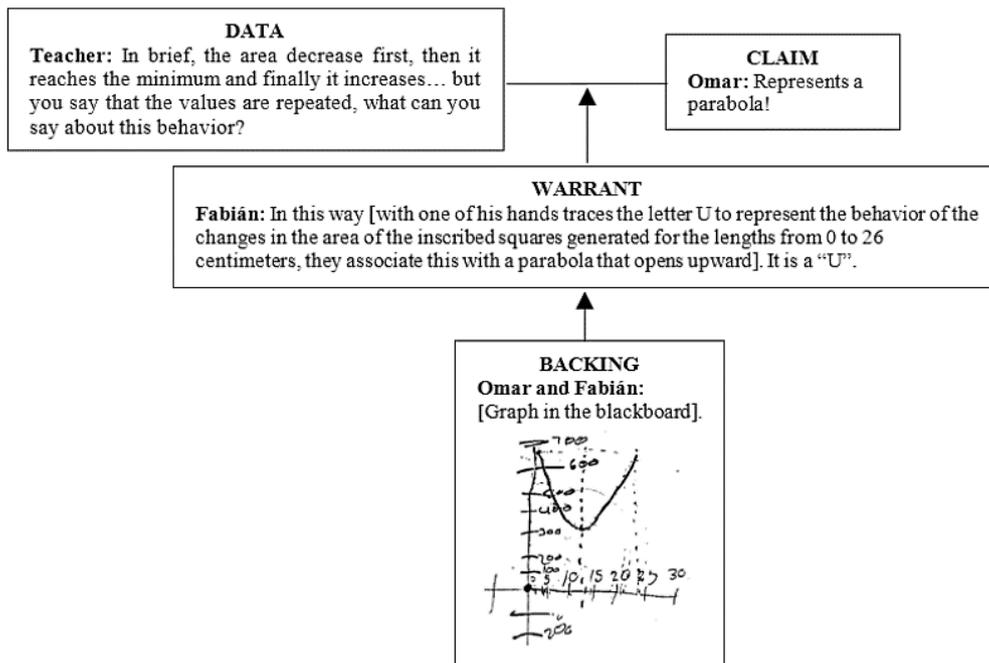
Figure 12. Argument 7. The analyzed variation is represented by a parabola
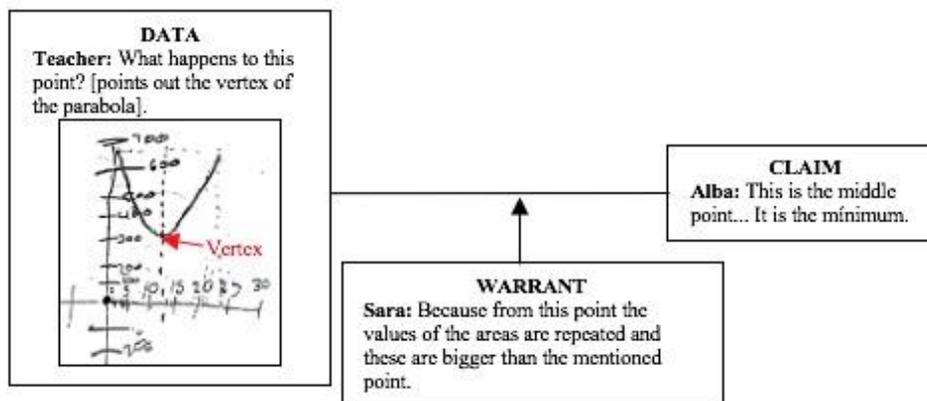
Figure 13. Argument 8. Analysis of the vertex of the parabola

Classroom Mathematical Practices

Two classroom mathematical practices were detected in the common mathematical activities that students carried out around the six normative ways: 1) analyze and quantify what changes in a variational situation; and 2) generate and interpret representations about how these variable magnitudes change in a variational situation.

The next sections document the two mathematical practices

Mathematical practice 1: Analyze and quantify what changes in a variational situation

The identification of this practice was based on the recognition of an invariant mathematical activity of the students in the first three normative ways of reasoning (NWR1, NWR2 and NWR3). The collective activity was characterized by the analysis of the magnitudes that vary in the situation and the quantification of the variation of these magnitudes. So, the practice consisted in the use of procedures and the establishment of mathematical relationships –for example, the PT, $\ell = c$, $c^2 = \ell^2 = A$ and the DA– and of properties to calculate and compare the measurements of the areas of the inscribed squares generated with specific values of $\overline{AE}$, and the mathematical reasoning of the students that were supported by numerical (e.g. Figure 4) and geometrical (e.g. Figure 3) arguments. Therefore, the intuitive arguments based on the context of the situation were the most common arguments in this practice.

Mathematical practice 2: Generate and interpret representations about how these variable magnitudes change in a variational situation

The identification of this practice was based on the common activity of the students in the last three normative ways of reasoning (NWR4, NWR5 and NWR6). The orientation of the reasoning in this activity focused on the generation of ways to represent the relation of correspondence between the variable magnitudes in the situation in order to analyze how the changes came about in the measurement of the area of the inscribed square in terms of the variation of the measure of the segment $\overline{AE}$ and to determine the dimensions of the square with minimum area. In this way, the students used tabular (Figure 10), graphic (Figure 12) and algebraic (Figure 9) systems of representation to describe and interpret the increasing and decreasing behavior of the measurement of the inscribed square. This behavior corresponds to the characteristic variational behavior of the quadratic functions. Reasoning was supported by mathematical arguments (e.g. Figure 11 and Figure 13) based on the representation of the quadratic behavior of the variables in this practice; therefore, it was characterized by the use of more formal arguments associated with the concept of quadratic function.

Discussion and Conclusion

Two mathematical practices that supported the collective mathematical progress in the students were recognized: 1) analyze and quantify what changes in a variational situation that involved the quadratic behavior among variables; and 2) generate and interpret representations of how the variable magnitudes change in a variational situation using ideas and representations associated with the quadratic function.

The first mathematical practice was constituted by the normative ways of reasoning (NWR1, NWR2 and NWR3) that made the students analyze what changes in the variational situation and quantify the changes in the identified variables. In this practice, the numerical and geometrical arguments are the most used arguments to generate conclusions about the changes in one of the variable magnitudes (measure of the segment AE) and the resulting change in the other magnitude (measurement of the area of the inscribed square). This means that this practice is characterized by an intuitive approach to the mathematization of the situation through arguments based on the specific context of the situation.

Based on the results of the first practice, the mathematical process was directed to determine the minimum value of the measurement of the area through normative ways of reasoning (NWR4, NWR5 and NWR6); these ways emerged to establish some relation of correspondence between the values of the variable magnitudes and to interpret the increasing and decreasing behavior of these values in a more systematic way. These normative ways constituted the second mathematical practice. Therefore, this practice is characterized by the construction and interpretation of tabular, graphic and algebraic representations to generate argumentations about the behavior of the changes of the dependent variable of the situation (measurement of the area of the inscribed square) in terms of the independent variable (measure of $\overline{AE}$). For example, the use of numerical and graphical representations of the values of the variables supported the reasoning of the collective to argue how the measurement of the area of the inscribed square increases or decreases and how much, and to determine the minimum value of the area and justify if this value was right or not.

The evolution of these mathematical practices was influenced by three interconnected elements that generated a model (Figure 14) of mathematical progress in terms of the six normative way identified: time, structure and mathematics. *Time* makes reference to the moment where the mathematical practices emerge in the classroom. The first mathematical practice emerges in sessions two and three with the development of NWR1, NWR2 and NWR3. The second mathematical practice emerges in sessions three and four with NWR4, NWR5 and NWR6. Figure 14 exemplifies the element time by providing the normative ways that emerged in the same session in a linear manner.

The *structure* is associated to the connection established between two mathematical practices. This means that the mathematical practices are connected with each other forming a net or a structure of networks of mathematical ideas or normative ways of reasoning (Stephan & Rasmussen, 2002). The structure of the identified network associated to the quadratic function goes from the more intuitive to the more formal normative ways of reasoning. For example, NWR1 and NWR2 emerged simultaneously and integrally in the discourse of the students based on geometrical arguments (Figure 3 and Figure 5) in the variational situation. This emergence allowed them to establish mathematical procedures and relations to quantify and analyze the

measurements of the areas of the inscribed squares generated by specific measures of the segment AE. NWR3 and NWR4 are based on the first two normative ways to generate new representations of the variation as part of its arguments, explaining the variation in this way, first in intuitive ways as measurements of the area repeated from the minimum area (Figure 8) and then in a more formal way as a relation of correspondence between the variable magnitudes represented in an algebraic form that considered the PT or the DA (Figure 9). NWR5 and NWR6 were based on NWR4 in order to generate arguments based on tabular (Figure 10) and graphical (Figure 12) representations of the relation of correspondence between the variable magnitudes. These normative ways facilitated the interpretation and description of the behavior of the changes in the analyzed variational situation, such as the recognition of increasing and decreasing values (Figure 11) and the identification of the minimum value (Figure 13). Figure 14 represents the structure element with arrows.
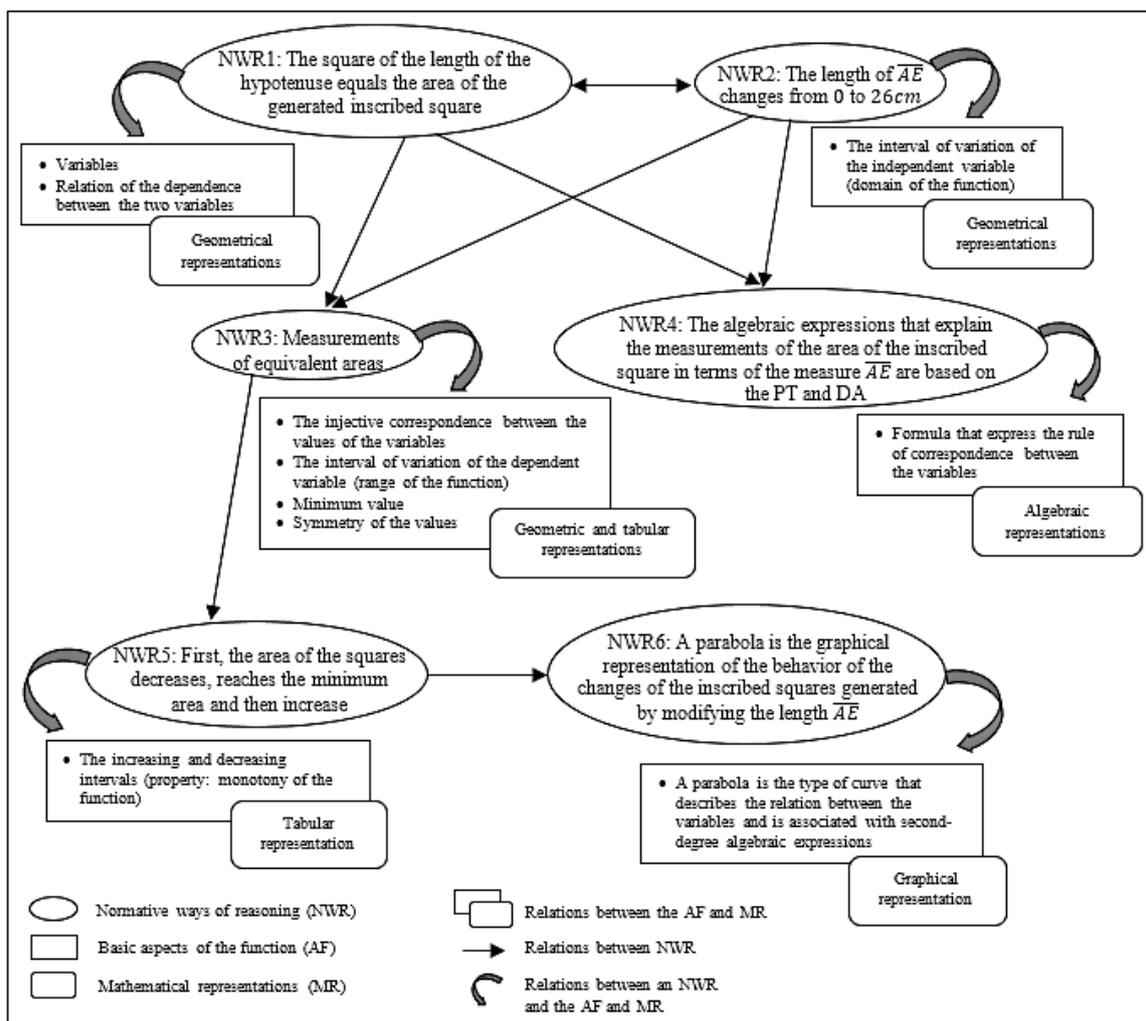
Figure 14. Collective mathematical progress in the treatment of the quadratic function

The *mathematics* in this mathematical practices is characterized by the mathematical representations and the basic aspects of the quadratic function that were behind the use of this function as a tool for the mathematization of this variational situation. In the first practice, the geometrical (NWR1 and NWR2) and tabular (NWR3) representations were used to promote the identification of the variable magnitudes, the relation of dependence among them, the interval of variation for the independent and dependent variables, the minimum value, the symmetry of the values of the dependent variable and to express the relation of correspondence between the values of the variables as ordered pairs.

In the second practice, algebraic (NWR4), tabular (NWR5) and graphical (NWR6) representations were used. The algebraic representations engaged the students to use variables such as "x" for the measurement of $\overline{AE}$ and "$A(x)$" for the measurement of the area of the inscribed square to express the formula of the rule of correspondence between the variable magnitudes in the variational situation. The tabular and graphical representations facilitated the recognition of the intervals of the increase or the decrease of the values and that

the type of curve that describes the relation between the magnitudes is the parabola and connecting them with algebraic expressions of second degree. This is why it is said that a mathematical progress is achieved in this collective. Figure 14 relates the mathematical representations and the basic aspects of the quadratic function present in the production of each normative way.

The model of the collective mathematical progress (Figure 14) shows the emergence of the mathematical practices during the treatment of the quadratic function; it did not follow a sequence with respect to time and structure. This result was also reported in Stephan and Rasmussen (2002) when analyzing the mathematical practices established in a Differential Equations course. This contrasts with the mathematical practices established in a community of students of first grade of elementary school (Cobb et al., 2001) where the productions followed a sequence with respect to time and structure. These results show that the emergence of classroom mathematical practices does not always follow a sequence in time and structure.

While recognizing that the mathematics put into play during the treatment of the quadratic function in this study has already been addressed in other investigations (e.g. Dolores 2013; Villa-Ochoa, 2012; Thompson & Carlson, 2017), this research also reports on the mathematics related with the production of classroom mathematical practices. It identifies the progress of the students in the recognition of the basic aspects of the quadratic function and their different representations by connecting and working with previous knowledge in Plane Geometry (geometrical figures and their areas), Arithmetic (numbers and mathematical operations), Algebra (variables and mathematical expressions) and Analytic Geometry (coordinate system and parabola).

Finally, the contribution of this work is established on two levels. On a micro level, the reports of the mathematical practices established in the treatment of the quadratic function in an Introductory Calculus course in middle high school that support the local and immediate situation (Cobb, 2000) of the collective mathematical progress. On a macro level, a model of the collective mathematical progress focused on three interconnected elements: time, structure and mathematics. These elements are present in the evolution of the mathematical practices. The proposed model has the quality of being explicative and the ability to adjust to interact with other students. This means, that the model documents a path for learning important ideas related to the quadratic function.

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