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Geometric Transformations and Symmetry in Primary and Secondary Education: A Review of Themes, Media, and Theoretical Frames from 1990 to Present

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Abstract

There is need for increased focus on geometric transformations and symmetry in primary and secondary education, as well as increased research to support learning and teaching of geometric transformations. To identify directions for future research and teaching, we set out to map the research that has already been conducted and to identify key areas of focus and opportunity going forward. Toward these goals, this systematic review examines 62 peer-reviewed articles on teaching and learning about 2D geometric transformations and symmetry since 1990. To guide our review, we use the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA) protocol. The review explores the research in terms of: (a) how students learn about transformations, (b) how teaching about transformations has been conceptualized, (c) how media have been leveraged to support learning about transformations, and (d) which theoretical frames have been leveraged and how have those frames shifted over time. Discussion and conclusions consider key areas of growth for the field going forward to better support teachers and students learning about symmetry and transformations.

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Introduction

There is need for increased focus on geometric transformations and symmetry in primary and secondary education (Goldenberg, Cuoco, & Mark, 1998; Driscoll, 2007) as well as increased research to support learning and teaching of geometric transformations and symmetry (Gutiérrez et al., 2016). Geometric transformations and symmetry provide teachers and educators opportunities for culturally responsive mathematics experience. These opportunities can demystify complex mathematical ideas and make them meaningful for students from diverse backgrounds by providing scenarios that navigate and describe their world (e.g., Kalinec-Craig et al., 2019). Historically, symmetry has connections with multiple disciplines, including physics and biology, providing students with access to big ideas in geometry that cut across other mathematical areas and STEM disciplines (Stewart, 2008; Villarroel et al., 2023).

A systematic review of geometric transformations and symmetry is thus important for multiple reasons. First, although current initiatives call for a greater focus on geometric constructs in mathematics education to broaden the conception of mathematics encountered in schools, to increase students' sense of its relevance to their lives and cultures, and to prepare students for their futures with mathematics (Goldenberg, Cuoco, & Mark, 1998; Mullis et al., 2021; Crosswhite et al., 1989; NCTM, 2000; OECD, 2018), mathematics education is still coming to terms with a historical hierarchy that has positioned algebraic constructs as more essential and robust than geometric constructs (refer to Goldenberg et al., 1998, and Driscoll, 2007, for extended arguments about the dangers of this hierarchy in mathematics education; refer to Atiyah, 1982, 2003, for an extended argument against the hierarchy directed at mathematicians). Second, enhancing our understanding of the gaps, strengths, and evolution of research can shed light on new directions for future studies, and highlight possibilities for new connections with innovations in research approaches and with socially significant ways of knowing.

Researchers and educators have long recognized the significance of geometric transformations and symmetry in mathematics education (Freudenthal, 1971; Usiskin, 1972; Gutiérrez & Boero, 2006). The importance of these concepts gained further prominence and support, however, with the launch of the NCTM's Standards Project in 1989 (Crosswhite et al., 1989; NCTM, 2000). The NCTM's Standards Project emphasized the inclusion of symmetry and transformations in the mathematics curriculum, ultimately enriching students' learning experiences by promoting spatial visualization and reasoning. Aligning with these calls for increased curricular emphasis on symmetry and transformations, three recent reports have highlighted the importance of research in these areas:

(a) the International Congress on Mathematics Education ICME-13 reports (Sinclair et al., 2016, 2017),

(b) the Second Handbook of Research on the Psychology of Mathematics Education (Gutiérrez et al., 2016), and

(c) a network analysis focused on "spatial reasoning" by Bruce and colleagues (2017).

All three reports indicate a need for more research focused on symmetry and transformations. Gutiérrez et al. (2016) explicitly state, for example, that transformational geometry is an under-represented research topic among the Psychology of Mathematics Education (PME) research community.

Given this gap in the literature, we set out to examine the research on transformations and symmetry from 1990 to the present. We have bounded this study from 1990 to acknowledge the significance of the 1989 NCTM standards as a watershed and also because new media became more readily available during this period for use by educators (including dynamic geometry and programmable environments), which has led researchers to rethink how learners might conceptually understand geometric transformations and situate their research with different and innovative theoretical lenses. Our goal is to synthesize the research on the ways students learn and teachers teach about 2D geometric transformations and symmetry in order to promote the creation of future possibilities in the development of transformational geometry research in mathematics education. More specifically, our review addresses the following questions:

1. What has research demonstrated in terms of how students understand geometric transformations and symmetry?
2. What has research demonstrated in terms of how to teach about geometric transformations and symmetry?
3. How have media been used in research on teaching and learning of geometric transformations and symmetry?
4. Which theoretical frames have been leveraged in this research, and how have those frames shifted over time?

Methodology for the Systematic Literature Review

Search Strategies

To answer the proposed research questions, we conducted a systematic review drawing on the PRISMA (Preferred Reporting Items for Systematic Reviews and Meta-Analyses) protocol as outlined by Moher et al. (2015) with a specific focus on geometric transformations. While PRISMA has its origins in health-care disciplines, the Protocol has "been widely endorsed and adopted, as evidenced by its co-publication in multiple journals, citation in over 60000 reports (Scopus, August 2020), endorsement from almost 200 journals and systematic review organisations, and adoption in various disciplines" (Page et al., 2021, p. 2).

We searched Academic Search Complete (EBSCO), ERIC, SCOPUS, and Web of Science (WOS) electronic databases for peer-reviewed English language academic articles. The keywords geometric transformation, geometry transformation, translation, rotation, reflection, and symmetry were selected based on the concepts traditionally linked to geometric transformations in various study programs in primary and secondary education (e.g., Mullis et al, 2021; NCTM, 2000).

Titles and abstracts of the selected studies were screened according to the following criteria:

(a) there was a focus on the teaching and learning of 2D geometric transformations;

(b) learning theories were discussed to understand how students or teachers think about geometric transformations in primary and secondary education; and

(c) there was sufficient description of the environments, tools, and activities in the study.

Ultimately, we arrived at a selection of sixty-two articles for inclusion in the literature review.

Search, Screening, and Analysis Process

The search process was carried out in three phases, following guidelines for systematic review set out in Moher et al. (2015) and Rowan et al. (2021), as presented in Figure 1. First, *identification* presents the number of raw hits obtained in all databases (899). Hits were discarded if the terms *translation* and *reflection* were linked to their colloquial meanings, the first focusing on *process of change* (e.g., high school students *translating* algebraic word problems into mathematical equations), and the second associated with its connotation as the production of ideas as a result of a thinking process (e.g., mathematics teachers' *reflections* on using technology). Next, notably, the term *rotation* yielded results related to geometry and spatial reasoning focusing on mental rotations of 3D objects, both physical and virtual. Our focus on 2D shapes led to the removal of these articles from our final count.

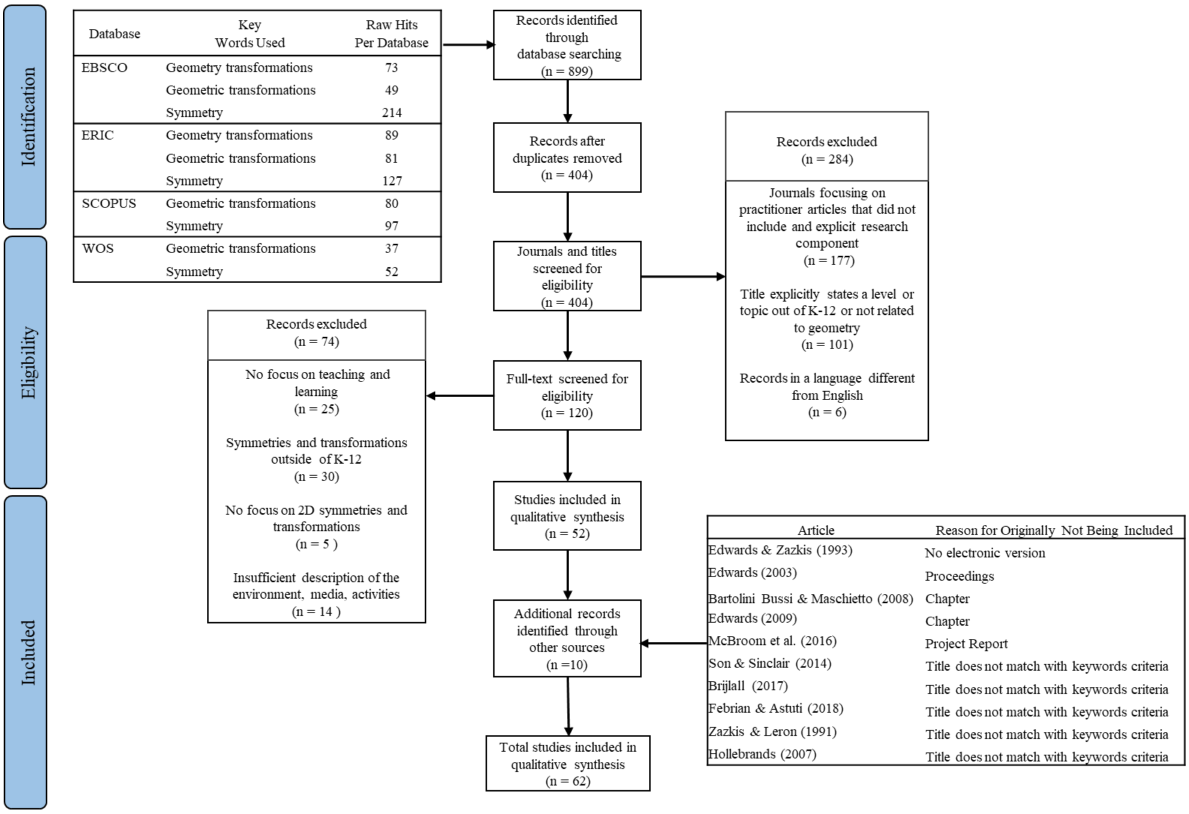

Figure 1. PRISMA Diagram Detailing Identification, Eligibility, and Inclusion

Second, we filtered for *eligibility.* After filtering duplicate entries (articles repeated in each search) we screened 404 articles' titles and journals names. We narrowed our selection by excluding articles with the following elements: (a) journals focusing on practitioner articles that did not include an explicit research component; (b) articles titles explicitly stating a level or topic outside of primary or secondary levels, or not related to 2D geometry; and (c) records in a language other than English.

Third, after this filtering, we conducted a full-text review of the remaining 120 hits. As mentioned, in order to focus the review, we narrowed our final selection of articles to include these elements: (a) there was a focus on the teaching and learning of 2D geometric transformations; (b) learning theories were discussed to understand how students or teachers think about geometric transformations in primary and secondary education; and (c) there was sufficient description of the environments, tools, and activities in the study. Articles excluded may have included one or two of the elements but not all three. Ultimately, the search process yielded 62 articles. While reviewing these articles to identify general trends, we noticed that certain references were repeatedly mentioned which were not identified during the database review. This suggested that those papers are essential for a thorough scan of the field. According to Brown (2017), who cites Walsh and Downe (2005), in order to overcome the limitation of inadequate qualitative accounts, "search strategies" should incorporate "traditional methods of reviewing" such as "back-tracking of references" (p. 1150). In doing so, ten additional records were added based on citations found in the thirty-nine articles identified in the search, for a total of 62 articles. At least two authors examined each of the 62 articles in terms of each of the four research questions.

Results

The results are presented in four main sections in relation to each of our four research questions: (a) how students understand transformations, (b) how teaching about transformations has been conceptualized, (c) how media have been leveraged to support learning about transformations, and (d) which theoretical frames have been leveraged in this research and how have those frames shifted over time.

How Students Understand Geometric Transformations

A predominant theme focused on helping students move their understandings of transformations from everyday physical experiences to more formal mathematical understandings (e.g., Faggiano et al., 2018; Gülkılık et al., 2015; Panorkou & Maloney, 2015). The research generally presented this as a transition from "motion" understandings to "mapping" understandings. More specifically, pervasive across the reviewed articles was the claim that learners understand geometric transformations in two ways: initially through a motion view that might potentially be integrated later with a mapping view (Edwards, 2003, 2009; Edwards & Zazkis, 1993; Hollebrands, 2003; Yanik & Flores, 2009; Xistouri et al., 2014). This claim was mirrored across radically diverse theoretical discourses, for example in relation to Sfard's (1991) operational/structural duality, Arnon et al.'s (2014) action and object perspective, and Tall et al.'s (1981) process and object duality. Essentially, according to the literature, the majority of people of all ages understand transformations as motions (e.g., Edwards, 2003, 2009; Hollebrands, 2003; Yanik & Flores, 2009). Learners operating within a motion-based perspective may consider the plane as a background and view a transformation as manipulation of a geometric figure on this background. Analogies (such as using a moving elevator to describe translation) and physical manipulation can make the concept of geometric transformation accessible to learners, but these analogies can also be a source of misconceptions (Yanik, 2014). For example, someone who holds only a motion view may consider physical force necessary for a transformation to occur (Yanik, 2011, 2014). Similarly, the motion view overlooks the important mathematical properties and relationships associated with each transformation. For instance, if one understands reflection vaguely as a "flip," it is possible one would not consider the properties of perpendicularity and equidistance to the axis of reflection. Finally, the motion view can be cognitively limiting, leaving learners only able to carry out operational procedures such as move, flip, and turn (Yanik, 2014).

The motion view in relation to the mapping view was clarified by Edwards (2003), who contrasted how learners and mathematicians viewed the geometric plane. Edwards suggested that mathematicians conceptualize the geometric plane metaphorically, formally, and abstractly as a set of points in space, while students often understand the plane primarily intuitively through their embodied personal experience. In summary, research on learning about geometric transformations shared a crucial focus on the transition between students' everyday understandings to more formal mathematical understandings. Although different theories utilized different theoretical constructs to explore this duality (e.g., conception/concept; Balacheff, 2013; process/object; Hollebrands, 2003, 2007; informal/formal; Brijlall, 2017; motion/mapping; Edwards, 2003; Yanik, 2011), all of them considered the everyday experience as a starting point from which students can produce formal understandings.

How Teaching about Transformations has been Conceptualized

Research within this theme highlighted a dialectical interaction between research on learning trajectories/learning progressions, mainly linked to individual cognition, and explorations of integrated teaching approaches, often highlighting socio-cultural and communicative dimensions. We therefore delve into this dialectical interaction, acknowledging the interacting significance and potential for further advancement of both areas of research.

Scholars often used the term *learning trajectories*, sometimes interchangeably with the term *learning progressions* (NCTM, 2014), to describe "conjectures about both a possible learning route" (Clements & Sarama, 2004, p. 82) and the development of a sequenced selection of instructional tasks (Battista, 2011; Clements & Sarama, 2004). For example, Fife et al. (2019) stated that learning progressions, validated through empirical research, could inform teaching and assessment. In contrast, other more socially integrated teaching approaches, mainly linked to learning in community, focused on interactions within the environment and community (Edwards, 1997; Gadanidis et al., 2018; Jacobson & Lehrer, 2000).

A challenge in teaching and learning about transformations was understanding how learners moved from the motion view to the mapping view (often with hesitation and vacillation) as outlined in earlier sections. It was this *in-between-ness* in the learning trajectory, as described by Sinclair et al. (2016), that researchers struggled to understand. Sinclair and Moss (2012) presented the notion that, in observing geometric discourse, the trajectory between levels of motion and mapping does not transpire "in smooth linear transition from one level to another; rather, it involves oscillating between the old and new forms of discourse, resulting in intermediary hybrid forms of geometric communication" (p. 43). Based on Sinclair and Moss's statement, scholars in mathematics education emphasized "the need to study the transition phases in the progress of geometrical concept formation" (Sinclair et al., 2016, p. 696).

Researchers have examined learning trajectories in order to determine students' transition from motion to mapping, while also acknowledging that students do not follow a linear trajectory (Law, 1991). While acknowledging this non-linearity and variation among individual students, however, promise was seen in learning trajectory/progression approaches, as evidenced by publications such as Fife et al. (2019), which devised a learning progression to support instruction and assessment more broadly. Some cross-cultural research, for example, set out to analyze and compare the cultural application of learning progressions in textbooks between France and Norway (Pepin et al., 2013) and between Japan and the UK (Takeuchi & Shinno, 2020). Overall, there appeared to be a tug-of-war between the very real need to document and understand learning phases, often for instructional and evaluative purposes (Fife et al., 2019), and the need to attend to the complex generativity that emerges in a classroom. This tension was aptly described by Lehrer et al. (1998) when they set out to test and ultimately push back against van Hiele's (1957) learning model. As Lehrer et al. explain, "classroom ecologies are not given but are continuously generated and achieved by participants so that conceptual change in reasoning about space (or anything else) is apt to depend on the dynamic qualities of interactions among students, teachers, tasks, tools, conversations and so on" (p. 163).

Based on the literature, however, learning trajectories serve researchers and teachers well, not only as a tool to assess learning but also to assist in developing a research-based blueprint for teaching and learning. Conversely, for some authors, the way in which mathematics learning was perceived through the lens of a learning trajectory/progression meant that opportunities to interrupt traditional thoughts about learning and to consider new possibilities may be overlooked. It was suggested by some that conducting research and curricular design in more integrated ways could permit researchers to observe and consider how learning about transformations sits within the space of big mathematical ideas. Gadanidis et al. (2018) explained, "The traditional - the easy - way of engaging young children with big mathematical ideas is to fragment the ideas to an extent that the mathematical structure is lost" (p. 34).

Teachers and researchers, however, often rely on learning trajectories to help them consider what and how to teach. The challenge, as identified by some, was that in order to design rich, more integrated learning experiences, teachers needed to possess substantial mathematical knowledge to envision creative and unique approaches for their students to experience curriculum. Given that many of the pre-service mathematics teachers featured in the studies did not themselves have a strong understanding of the abstract mapping ideas inherent in the mathematical view of transformations (e.g., Avcu & Çetinkaya, 2019; Hegg et al., 2018; Son & Sinclair, 2010; Yanik, 2009, 2011), they required support in developing and implementing integrated learning activities. Another identified conundrum for teachers was assessment. Learning trajectories provided a structure and focus for teachers and researchers to determine what to assess and what learning should precede other learning. In more integrated approaches, which were often more generative in nature, this was not always easily determined, particularly for those teachers who did not possess deep knowledge of the discipline.

How Media have been Leveraged to Support Learning about Transformations

With the development of more sophisticated digital technologies, researchers considered the affordances of particular media, and adopted and adapted them for use in a variety of contexts. The use of different media enabled researchers to investigate a wide spectrum of questions from the cognitive affordances of dynamic geometry environments to the sociocultural contexts that promoted the relationships between teacher, students, and media. The literature review identified four main categories of media use: (1) dynamic geometry environments (e.g., GeoGebra, Cabri, Geometer's sketchpad); (2) paper and pencil written problem-solving activities that did not require other media (e.g., multiple-choice and written tests and activities); (3) programmable environments/digital tools where the user gave a series of mathematical rules to be executed by one or more computer agents (e.g., Logo and related programming environments) or game avatars (e.g., in educational video games); and (4) physical tools (e.g., patty paper, rulers, compasses, geodreiecks, and linkages). Four studies did not articulate an explicit use of media (Fife et al., 2019; Hāwera & Taylor, 2014; Pepin et al., 2013; Takeuchi & Shinno, 2020). The overall trend in Figure 2 highlights a shift over the decades from a primary focus on programmable environments and digital tools to a more varied use of instructional media. Across the years research took a more balanced approach to the use of media, as the articles from the 2020s illustrate. Given the increasing integration of technology in geometric transformations and symmetry research over the decades, we believe this field offers a unique intersection of digital and tangible learning tools making this area a rich and dynamic field of study.

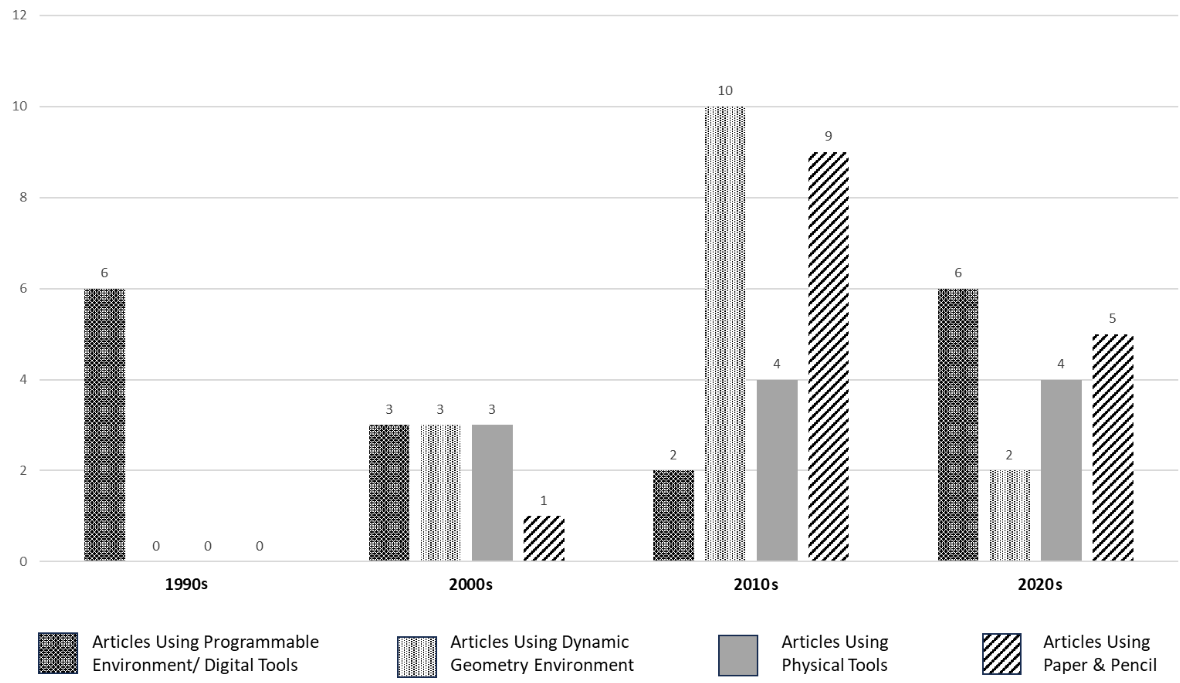

Figure 2. Distribution of Media since 1990

Dynamic Geometry Environments

The 2010s research (see Figure 2), in particular, saw intensive use of dynamic geometry environments that allowed students and teachers to explore and construct dynamic objects (Schwartz et al., 1993). According to the literature, dynamic geometry environments engaged students in constructing explanations and justifications about the properties of geometric transformations (Avcu & Çetinkaya, 2019; Hollebrands, 2003; Yao & Manouchehri, 2019) while offering specific additional affordances. Mainali and Heck (2017), Tatar et al. (2014), and Thangamani and Kwan Eu (2019) illustrated how dynamic geometry environments provided immediate feedback on the transitions among multiple representations (e.g., graphical and algebraic), facilitating the study of transformations as functions.

In addition, the dragging feature (direct and continuous manipulation of “hot spot” points) of dynamic geometry environments was a central focus for some researchers (e.g., Guven, 2012; Hollebrands, 2007), in that it was the main characteristic of geometric dynamism and a contrast to static, pencil-and-paper geometry. Research in symmetry and transformations focused on the “drag test” (confirming that a property of the diagram remains invariant under dragging), which fostered proof congruency skills in Euclidean geometry (Arzarello et al., 2002; Hollebrands, 2007). This also meant, however, that dragging as an opportunity for learners to explore continuous motion of objects/actions that might have linked geometric transformations with a learner’s prior physical experiences was a neglected affordance.

Paper-and-Pencil

Research using only paper-and-pencil was underrepresented during the first two decades but increased in the 2010s and 2020s (Figure 2). This medium allowed for assessment of individual performance but also offered a window into the conceptions of students (Yanik, 2014). The curriculum commonly guided the design of paper-and-pencil tasks, where investigations frequently addressed proofs of congruence (Hegg et al., 2018; Portnoy, 2006) and the inclusion of problems that required the visualization of images (e.g., drawings or points on the Cartesian plane) (DeJarnette et al., 2016; Mhlolo & Shafer, 2014; Son & Sinclair, 2010; Yanik, 2011, 2014). Although paper-and-pencil and dynamic geometry environments can both focus attention on understanding the shift from motion to mapping through the development of proofs and algebraic properties of transformations using visualizations, a crucial difference between the two was in how the immediate feedback features of dynamic geometry empowered students’ understanding.

Programmable Environments/Digital Tools

A focus on programmable environments spanned research on individual cognition and the design of environments to scaffold cognition around transformations. We surmise that the dissemination of Logo variants through the early 1990s, influenced by constructionist ideas (Papert, 1980) and a strong cognitivist tradition that researchers had at the time, led to this boom. According to Edwards (1991), computational environments “embod[y] some area of mathematics so that the mathematical objects and operations behave according to their formal definitions” (p. 123). Indeed, these environments helped students build initial understandings of geometric transformations (Hoyles & Healy, 1997; Zazkis & Leron, 1991). In the reviewed works, we found that a central characteristic of programmable environments was the use of parameterized transformation commands to apply to a shape (e.g., “Slide 10 -20” is a translation of the coordinate plane taking the origin to the point (10, -20)). A vital aspect of these investigations is that learners were supported in attending to complex or advanced mathematical ideas in conjunction with computational thinking (e.g., the mapping conception, based on Klein’s *Erlangen* program, Edwards, 1991, 2003, 2009, or the group theory perspective that Gadanidis et al., 2018, studied). The emphasis of the research focused on everyday and embodied experiences as valid resources and starting points in understanding the mapping view of geometric transformations. In this research, computational thinking offered unique affordances and an accessible, agentive approach to mathematical learning (Gadanidis et al., 2018; Hoyles & Noss, 2020) through the enactment of algorithms, abstractions, decomposition, pattern recognition, and generalizations that are "deeply mathematical" (Hoyles & Noss, 2020, video time 24:38).

We encountered two papers in our review that focused specifically on the use of digital games for learning. Sedig (2008) developed and studied an environment that supported learners in using the language of transformations to operate on tangrams. Transformations mediated the interaction with the tangram pieces in a way analogous to that of informal programming. This suggests the potential at the intersection of games (or playfulness) and programming as an area for research on learning about transformations in the future. Authors (2023) recently published research on the ways in which a digital game that blended geometric transformations and computational thinking supported students' integration of everyday and formal concepts in ways aligned with Vygotsky's (1986) ideas about "the development of the child's spontaneous concepts" (p. 193).

Over the time span of this review, programmable environments were noted for their ability to offer immediate feedback to learners, supporting an embodied perspective that transformed student cognition and understanding about geometric transformations in terms of students' personal experience of motion.

Physical Tools

Physical tools frequently appeared in combination with other media. Whereas research on computational environments and digital tools has tended to focus on the study of individual relationships between media and participants, research using physical tools has tended toward a social and collective approach. The studies that included physical tools often investigated the cultural mediation of tools as learners and teachers work together (Bartolini Bussi & Mariotti, 2008). The articles we analyzed often incorporated digital media (dynamic geometry environments) in combination with physical media (paper-and-pencil, geometry tools). Faggiano et al. (2018) stated that the use of a combination of media meant that a "synergic action will develop in such a way that each activity boosts the learning potential of all the others" (p. 1173), emphasizing how each medium played a part in assisting understanding through different affordances, and how the synergies between media fostered emergent geometrical learning (Authors, 2021) as highlighted in a special issue of *Digital Experiences in Mathematics Education* (Mariotti & Montone, 2021; Nemirovsky & Sinclair, 2020).

Which Theoretical Frames have been Leveraged and how have those Frames Shifted Over Time

We analyzed the theoretical frames employed by the articles in terms of the learning discourses map authored by Davis and Francis (2023), which synthesizes approximately nine hundred theoretical frameworks in education, further supported by the extensive work of Davis, Sumara, and Luce-Kapler (2015), who delve into the conceptualization and significance of research epistemologies and learning theories within the classroom. In this step, we engaged in a three-stage verification process to cluster the articles (Miles et al., 2020) according to their theoretical frame. This process began with a comprehensive examination of the learning discourses and other reviews on geometry education (Bruce et al., 2017; Gutiérrez et al., 2016; Sinclair et al., 2016, 2017. Articles were then grouped using the following criteria: (1) articles that explicitly mentioned the use of the same theory were placed together as a frame; (2) articles using different theories yet similar focus on aspects of learning or teaching were merged (e.g., Action-Process-Object-Schema, APOS Theory, and the conception-knowing-concept (cK¢) model were merged into a single frame); and (3) articles that used multiple theoretical foci were located in the frames that best fit the ideas presented—potentially being included in more than one frame.

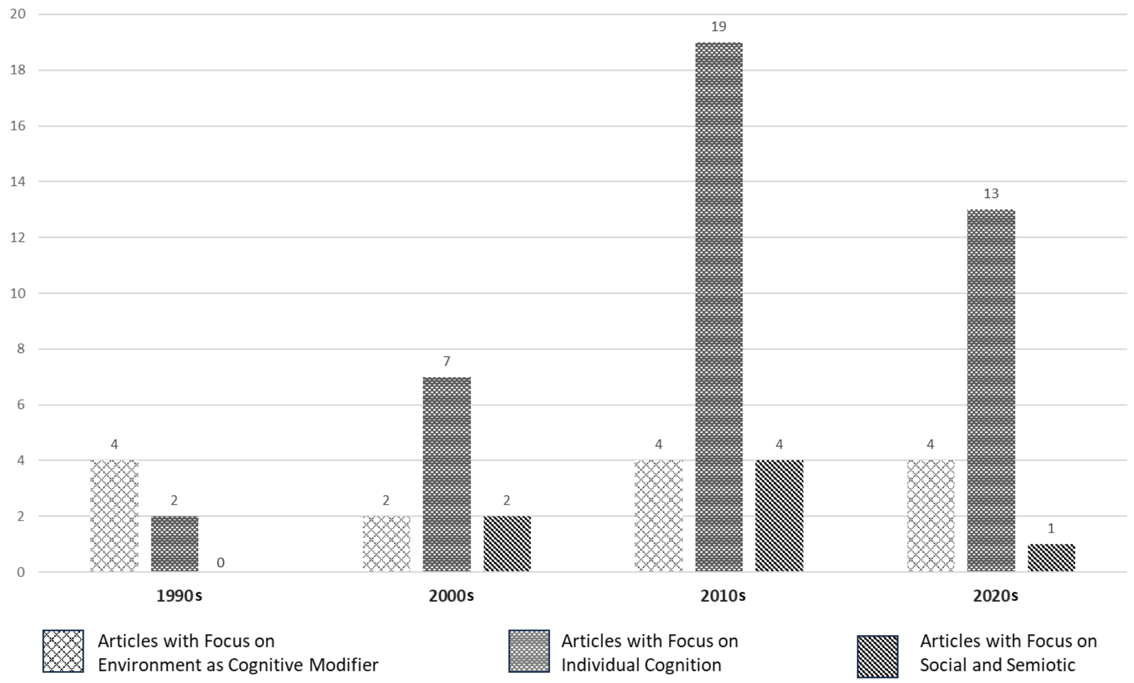

Figure 3. Distribution of Theoretical Frames since 1990

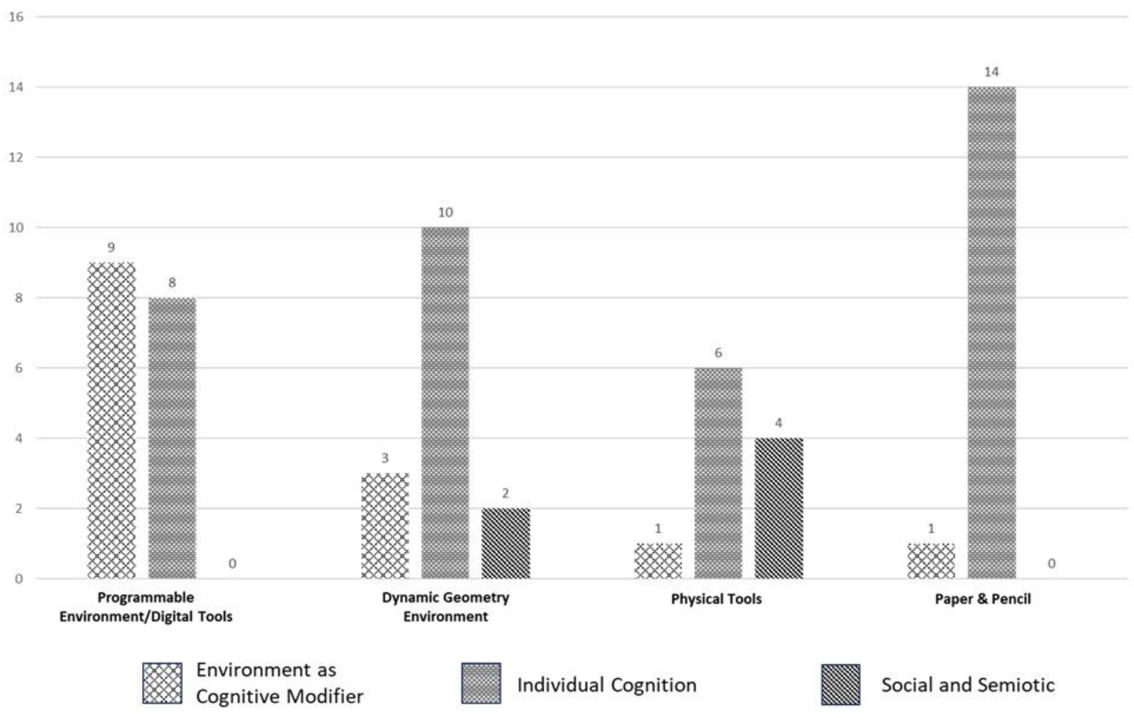

Figure 4. Distribution of Media by Theoretical Frames

The process then involved discussion and debate among all the authors about the placement of the articles into the frames, leading to several modifications of the placement of articles in each frame until a consensus was reached. The analysis yielded three overarching theoretical frames: (a) individual cognition, (b) learning environment as a cognitive modifier, and (c) social and semiotic approaches. Because the number of articles in the individual cognition frame was so large, we organized the articles within that frame in terms of three substantial foci.

Table 1 presents definitions for each frame. Table 2 presents the clustering of the 62 reviewed articles into the frames. Table 3 presents examples of coded excerpts in terms of the theoretical frames and media. Figure 3 presents the distribution of frames by decade. And Figure 4 presents the distribution of focal media by theoretical frame.

Table 1. Theoretical Frames

Frame	Description
Individual Cognition (Three Constiutent Foci Below)	The articles located in this frame often adopt a cognitive perspective associated with constructivist theories. These studies' view learning and knowledge as ongoing processes of change and the construction of conceptual relations from an individual perspective.
• Obstacles for Learning	These articles located explored obstacles that may impede a learner's ability to understand a concept, focussing on understanding learning processes from a cognitive or a conceptual point of view, often from the perspective of how learners took their experiential knowledge and applied it to mathematical learning.
• Processes and Levels of Learning	These articles frequently employed "levels of geometric thinking" model (e.g., van Hiele, 1957; Kuzniak, 2018) to explore disciplinary knowledge and learning pathways in order to analyze students' conceptual development and to assess learning.
• Teacher Learning and Autonomy	These articles addressed how teachers confronted rigid and alienating aspects of some models of schooling by agentively guiding their own learning experiences. The articles explored how educators took responsibility for their own inquiries, resources, and time management.
Learning Environment as Cognitive Modifier	The articles located in this frame focused on the analysis of what triggers change (Davis et al., 2015) when engaging in the exploration and construction of external objects (e.g., drawings, computer programs, 3D models, etcetera) coupled with experience and informal knowledge as a fundamental part of developing an understanding of geometric transformations.
Social and Semiotic Approaches	The articles located in this frame adopted participatory and culturally situated approaches often using the theory of semiotic mediation with a more obvious focus on teaching. Also important was the situated and distributed use of tools ranging from "vocabulary and discourse" (Davis et al., 2015, p. 157) to "artifacts, structures, and habits that enable, constrain, and otherwise channel what can be known and done" (p. 142).

Table 2. Articles Clustering Distribution by Code

Individual Cognition (n=41)			Learning Environment as Cognitive Modifier (n =14)	Social and Semiotic Appproaches (n=7)
(a) Learning Obstacles Focus (n=20)	**(b) Learning Processes and Paths (n=10)**	**(c) Teaching Frameworks (n=11)**		
(Edwards, 1991)	(Johnson-Gentile et al., 1994)	*(Portnoy et al., 2006)	(Edwards & Zazkis, 1993)	(Bartolini Bussi & Machietto, 2008)
(Sedig, 2008)	*(Yanik & Flores, 2009)	*(Yanik & Flores, 2009)	(Zazkis & Leron, 1991)	(Ng & Sinclair, 2015)
(Edwards, 2003)	(Guven, 2012)	*(Yanik, 2011)	(Edwards, 1997)	(Hawera & Taylor, 2005)
(Hollebrands, 2003)	*(Xistouri & Pitta Pantazi, 2014)	(Son & Sinclair, 2014)	(Hoyles & Healy, 1997)	(Yao et al., 2019)
*(Portnoy et al., 2006)	(Gulkilik et al., 2015)	(Tatar et al., 2014)	(Jacobson & Lehrer, 2000)	(Bansilal & Naidoo, 2012)
(Hollebrands, 2007)	(Fife et al., 2019)	(McBroom et al., 2016)	(Leikin et al., 2000)	(Breive, 2022)
(Edwards, 2009)	(Gutiérrez et al., 2021)	(Ferrarello et al., 2015)	(Panorkou & Maloney, 2015)	(Kalinec-Craig et al., 2019)
*(Yanik & Flores, 2009)	(Pepin et al., 2013)	(Febrian & Astuti, 2018)	(Fan et al., 2017)	
*(Yanik, 2011)	*(Dello Iacono & Ferrara Dentice, 2022)	(Thanganami & Kwan Eu, 2019)	(Uygun, 2020)	
(Mhlolo & Schafer, 2014)	(Sahara et al., 2024)	(Mainali & Heck, 2017)	(Faggiano et al., 2018)	
(Yanik, 2014)	(Mbusi & Luneta, 2023)	(Savaş & Köse, 2023)	(Gadanidis et al., 2018)	
*(Xistouri & Pitta Pantazi, 2014)	(Xu et al., 2023)	(Lai et al., 2023)	(Authors, 2023)	
(Dejarnette et al., 2016)	(Rawani et al., 2023)	(García-Lázaro & Martín-Nieto, 2023)	(Ng et al., 2023)	
(Brijlall, 2017)		(Pocalana et al., 2023)	(Cuturi et al., 2023)	
(Hegg et al., 2018)				
(Avcu & Cetinkaya, 2019)				
*(Dello Iacono & Ferrara Dentice, 2022)				
(Takeuchi & Shinno, 2020)				
(Hershkovitz et al., 2024)				
(Villarroel et al., 2023)				

Note: Articles marked with * share two or more subcodes. The count of these articles is part of the first subcode.

Table 3. Examples of Coded Excerpts for Theoretical Frames and Media

Study	Frame	Theoretical Approach	Coded Excerpt	Media
Guven (2012)	Individual Cognition (Learning Processes)	van Hiele Levels of Learning	"The result of covariance analysis showed that the experimental group outperformed the control group not only in academic achievement but also in levels of learning of transformation geometry." (Abstract, p. 364)	Dynamic Geometry Environment
Edwards (2009)	Individual Cognition (Obstacles)	Embodied Cognition	"In this chapter I describe the theory of embodied mathematics and utilize an embodied perspective on mathematical thinking to analyze data from students' experiences in learning transformation geometry." (p.29)	Programmable Environment
Son & Sinclair (2014)	Individual Cognition (Teachers Learning)	PCK	"This article originated with our interest in learning more about PCK in the specific domain of geometry —symmetry — with preservice teachers. We set out to investigate elementary preservice teachers' interpretation of and responses to a student's errors of reflective symmetry and its relationship to their knowledge." (p. 34)	Pencil and Paper
Gadanidis et al. (2018)	Environments as Cognitive Modifier	Vygotsky's zone of proximal development	"computer may play the role of a more knowledgeable other, which "is resonant with Vygotsky's (1978) theory that every stage of a child's development is characterized by an actual development level and a potential development level; the child may only be able to exploit his or her potential development level with help" (p. 32-33)	Programmable Environment
Bansilal & Naidoo (2012)	Social and Semiotic Approaches	Semiotic systems	"A semiotic system is characterised by a set of elementary signs, a set of rules for the production and transformation of signs and an underlying meaning structure deriving from the relationship between the signs within the system" (p. 29)	Physical Tools

Note: Subcoding *Obstacles* refers to the obstacles that may impede a learner's ability to understand a concept; subcoding *Learning Processes* refers to the processes and paths taken when learning about transformations; and subcoding *Teachers Learning refers to* teaching frameworks that support teacher learning and agency, at both pre-service and in-service level

Theoretical Frame: Individual Cognition

The largest portion of the articles (41 of 62) clustered within the theoretical frame of individual cognition. The dominance of this frame has increased over time. The raw number of articles within this frame as well as the percentage of articles within this frame relative to other frames has increased with each decade (see Figure 3). The articles located in this frame often adopt a cognitive perspective associated with constructivist theories. These studies' view learning and knowledge as ongoing processes of change and the construction of conceptual relations from an individual perspective. In terms of media, articles in this frame focused most heavily on dynamic geometry environments and pencil and paper with lesser emphasis on programmable environments and physical tools (see Figure 4). Within this overarching frame of individual cognition, three foci emerged:

(a) a focus on obstacles that may hinder a learner's comprehension of a concept;

(b) a focus on processes and approaches used when learning about transformations; and

(c) a focus on teaching frameworks that facilitate teacher learning and autonomy during pre-service and in-service education.

Articles that focused on obstacles that may impede a learner's ability to understand a concept, focused on understanding learning processes from a cognitive or a conceptual point of view, often from the perspective of how learners took their experiential knowledge and applied it to mathematical learning. Specifically, we mean the processes through which individual learners moved from intuitive understandings (e.g., transformations as motion) to more formalized mathematical understandings of concepts (e.g., transformations as functions, mappings of a plane to itself).

These articles tended to navigate two geometrical learning traditions. The first focuses on a formal vision of geometry, arguing that seeing transformations as actions that produce motion in concrete objects may lead to difficulties with congruency proofs (Brijlall, 2017; Hegg et al., 2018; Portnoy et al., 2006). The second focuses on a vision of geometry where the motion of transformed objects leads cognition and conceptual understandings toward a "mapping" conception of the plane (DeJarnette et al., 2016; Hollebrands, 2003, 2007; Yanik, 2011). According to Edwards (2003), the mapping conception based on Klein's Erlangen program "changed the focus of geometry from specific objects with concrete, visualizable referents (points, lines, planes, etc.), to notions of invariance, group theory, and mappings" (pp. 4-5). For instance, Yanik (2014) observed the transitions between experience in the physical world and the construction of a mapping point of view (e.g., motion as a vector).

There was an emphasis across the articles on the strengths and limitations of the informal resources that learners bring to the study of transformational geometry. In particular, on: (a) learners' naive, experienced-based knowledge as a set of resources for understanding transformations (Edwards, 2003), (b) learners' informal knowledge and visualization as potential impediments to understanding of formal mathematical concepts (Brijlall, 2017; Dello Iacono & Ferrara Dentice, 2022; Hegg et al., 2018; Mhlolo & Schäfer, 2014), and (c) learners' explanations of mathematical concepts as drawing upon everyday situations in a manner that can create tensions between informal and formal knowledge as stumbling blocks or scaffolds to understanding (e.g., pottery lessons to study reflections, DeJarnette et al., 2016).

Articles focused on processes and paths taken when learning about transformations, often employed a "levels of geometric thinking" model to represent disciplinary knowledge and learning pathways, to analyze students' conceptual development, and to assess learning. This research area emphasized, for example, that geometric models such as van Hiele's (1957) and Kuzniak's (2018) geometric workspaces provided a scaffold that framed learning progressions in levels, starting from the "easy" and working toward the progressively more "complicated" (symbolic) understanding (Gutiérrez et al., 2021; Guven, 2012; Johnson-Gentile et al., 1994; Xistouri et al., 2014). As another example, Gülkılık et al. (2015) used the Piere-Kieren model to highlight how students moved backward and forward as they built on experiential knowledge in constructing formal understanding (mappings and notations), with a key element being the use of manipulatives and multiple representations of transformations.

Articles focused on teaching frameworks that support teacher learning and agency, addressed how teachers confronted rigid and alienating aspects of some models of schooling by agentively guiding the learning experiences they created for their students. Two important frameworks, the original pedagogical content knowledge (PCK) framework (Shulman, 1986) and the subsequent adaptation in terms of the technological pedagogical content knowledge (TPACK) framework (Mishra & Koehler, 2006), referenced the distinct bodies of knowledge required for the complex act of teaching (Shulman, 1986). Key emphases in this teacher-focused area of the literature highlighted that:

(a) pre-service teachers' limited content and pedagogical knowledge impeded their ability to access practical knowledge (know-how) to deal with student mistakes and misconceptions because they themselves had not had opportunities to learn by making mistakes (Son & Sinclair, 2010);

(b) participation in implementation and reflection helped teachers understand geometry in new ways (Ferrarello et al., 2014); and

(c) training and support on the use of technology (Tatar et al., 2014) as well as the development of pedagogy was required in conjunction with mathematical knowledge (Leikin et al., 2000) and dynamic geometry environments (Mainali & Heck, 2017; McBroom et al., 2016).

Theoretical Frame: Learning Environment as Cognitive Modifier

A smaller cluster of articles (11 of 62), referred to a process in which interaction with the learning environment modified cognition. The articles located in this frame focused on the analysis of what triggers change (Davis et al., 2015) when engaging in the exploration and construction of external objects (e.g., computer programs, drawings, 3D models, etc.) coupled with experience and informal knowledge as a fundamental part of developing an understanding of geometric transformations. Most of the symmetry and geometric transformations research in this frame was constructionist in nature, where learners are seen to construct ideas "when they are actively engaged in making some type of external artifact" (Kafai & Resnick, 1996, p. 1). These constructionist discourses typically focused on programmable environments, or microworlds (Papert, 1987), but other media were also incorporated (Figure 4). What set these articles apart from those of the first frame was their focus on the adaptivity of the learning environment itself as a cognitive modifier. The number of articles in this frame has remained relatively constant over the decades (see Figure 3). Specific foci of the research emphasized:

(a) using digital media to advance learners' understandings of mathematical concepts related to transformations, often focusing on mapping of the plane as a point-set (e.g., Edwards & Zazkis, 1993; Panorkou & Maloney, 2015), reflection (e.g., Hoyles & Healy, 1997; Zazkis & Leron, 1991) and conceptual challenges (e.g., understanding rotation by an angle of 270° to the left as the same as 90° to the right);
(b) promoting mathematical strategies (e.g., using feedback to correct overgeneralizations) in computer environments (e.g., Edwards, 1991, 1997);
(c) developing problem-based pedagogies to assist students in using dynamic geometry to create proofs (Leikin et al., 2000) and in using auxiliary lines to solve geometric problems (Fan et al., 2017);
(d) supporting teachers in sensemaking discourse and argumentation (Jacobson & Lehrer, 2000); and
(e) creating mathematical and computational socially-shared experiences, where symmetry and transformations served as a conduit for the development of big ideas in mathematics (e.g., group theory, Gadanidis et al., 2018).

Theoretical Frame: Social and Semiotic Approaches

The smallest group of articles, only 7 of 62, clustered in relation to participatory and culturally situated approaches. Yet, articles with this focus have increased in prominence since 2010 (Figure 3). The articles located in this frame adopted participatory and culturally situated approaches often using the theory of semiotic mediation with a more explicit focus on teaching. Also important was the situated and distributed use of tools ranging from "vocabulary and discourse" (Davis et al., 2015, p. 157) to "artifacts, structures, and habits that enable, constrain, and otherwise channel what can be known and done" (p. 142). In particular, this frame highlighted the role of teachers in organizing and structuring learning and empowering students to respond to their own learning needs.

These articles clustered according to how learners made meaning of mathematical concepts through the use of signs and symbols. The teacher played an important role as a mediator, acting as a guide in meaning making (Ng & Sinclair, 2015). Most of the studies in this section focused on activities that employed artifacts because the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008) deals directly with the cultural mediation roles that artifacts (e.g., a compass or educational software) and teachers play in learning. In terms of media these studies have focused primarily on physical tools but have also included dynamic geometry environments (see Figure 4).

Key discursive emphases included: (a) the consequential interconnections across medium, student, and teacher; (b) the synergy that is created between physical and digital artifacts as an essential aspect of semiotic mediation and mathematical learning (Bartolini Bussi & Maschietto, 2008; Faggiano et al., 2018); (c) the substantive meaning in children's and teachers' gestures, diagrams, and discourses (Breive, 2022; Ng & Sinclair, 2015); and (d) conversions among registers of semiotic representation (Bansilal & Naidoo, 2012; Duval, 2017) that focused on developing relationships between visual and algebraic representations. Studies that explored culturally responsive meaning-making (Kalinec-Craig et al., 2019; Hāwera & Taylor, 2014) suggested new ways to consider how knowledge evolves through collective and cultural processes.

Discussion

The literature in the past three decades clustered distinctly, which suggests emergent and perhaps unintentional concentrations of emphasis in the field's collective research agenda as it has evolved over time (Figure 3). The distribution demonstrates that the theoretical focus on individual cognition has increasingly dominated research over the three decades, which indicates the consequential influence of cognitivist research, alongside the more recent emergence and increasing influence of social and semiotic perspectives.

In attempting to understand where the field stands today, this literature review has focused on studies from the early 1990s to the present day. The question posed by Moyer (1978) even prior to 1990 remains central: *How do students think about transformations?* Perhaps by rethinking methodological and conceptual approaches, we can come to a more complete understanding of how students learn and how teachers might teach about transformations. Our analysis highlights the potential of research that uses technology to support mathematical thinking and learning, tapping into visual forms of understanding that stand on their own next to algebraic ones. The emergence and use, for example, of dynamic geometry environments such as Cabri, Geometer's Sketchpad, and GeoGebra has enabled a shift toward representational infrastructures that depend less completely on "symbol strings" and algebraic formalisms. Several researchers' efforts in that direction exemplify this trend, such as Koile and Rubin's (2015) project exploring interactive visual representations of basic operations or the Kaput Center's SimCalc projects that focus on democratizing access to the mathematics of change and variation (Kaput Center for Research & Innovation in STEM Education, 2009).

Envisioning a Diverse Future: Integrating Cultural Contexts in Geometric Transformations and Symmetry Research

Most of the research we surveyed remains centered in Western perspectives on mathematics as a discipline. Our search yielded only one article that addressed Indigenous approaches (Hāwera & Taylor, 2014). Relatedly, the dearth of ethnomathematics articles identified in our search may be in part a result of their being classified in disciplines outside of education—such as anthropology. This highlights the need for more cross-cutting, transdisciplinary research in this area. In particular, the ethnomathematics literature represents important prospects for envisaging future and critical research, in that it considers how culturally relevant elements can support understanding of geometric transformations and fractals, often through computational environments (Barton, 1996; Eglash, 1999; Eglash et al., 2006; Eko, 2017; Prahmana & D'Ambrosio, 2020). This work can respond to mathematics education's long tradition of searching for culturally-rooted epistemologies (e.g., how cultures create and use mathematics in their everyday activities and Aboriginal perspectives, Sterenberg, 2013) in connection with formal mathematics as well as computational representations.

Moving Beyond Individual Cognition: Exploring the Social Aspects of Learning in Transformational Geometry Research

Current work in the domain of science education may also inform the use of learning trajectories that our review has shown as prevalent in mathematics education research on geometric transformations. Science educators are calling for scholars to consider the use of learning trajectories or learning progressions in new ways, and this may have ramifications for math educators. As Authors (2019) remark, “Learning progressions often obscure social processes of learning, instead approaching learning from an individual cognitive perspective” (p. xxx). Considering social aspects of learning may assist researchers in tracing “the progress of a practice as it is appropriated and transformed within a classroom community” (p. xxx). An increased emphasis on practices and the social dimension of knowing and learning could also provide entry points into culturally relevant teaching.

Integrating Symmetry, Transformations, and Computational Thinking: A Promising Research Landscape

Even though this review found a very limited number of articles (e.g., Gadanidis et al., 2018; Ng et al., 2023) that explicitly drew on the integration of symmetry and transformations with computational thinking, there were studies that used primitives and abstractions in programmable environments (e.g., Edwards, 1991; Hoyles & Healy, 1997), impacting the students’ understandings about the motion of geometric shapes under transformations. These investigations created a pathway that led to the integration of computing in the mathematics classroom. The recent resurgence of interest in computational thinking (diSessa, 2000; Li et al., 2020; Papert, 1980; Wing, 2006) within the field of mathematics and science education has reframed current research, indicating that this is an open area that would benefit from expansion.

A recent scoping review on computational thinking in K12 mathematics education (Hickmott et al., 2018) confirms this view. The authors state that “studies that explicitly linked the learning of mathematics concepts to computational thinking were uncommon” (p. 65). They recommend research that focuses on “effective ways of delivering integrated curricula” across domains, suggesting that “it is essential for research teams to draw on expertise which can bridge discipline silos without diminishing the importance of any constituent part” (p. 66). The overlap between computational and mathematical thinking is important, in that each can enhance the other, leading to a deepening understanding of mathematical ideas (Hoyles & Noss, 2020; Weintrop et al., 2016).

Classroom-Embedded Research: Participatory Designs to Foster Connections with Big Ideas in Geometric Transformations and Symmetry

Looking to the future, in taking up Moyer’s (1978) question, we suggest that the transformations research field should consider envisioning new ways of teaching, learning, and researching, including networked interdisciplinary research opportunities (Davis, Sumara, & Luce-Kapler, 2015). As part of this research, one question that could be asked is whether part of the difficulty learners experience in making sense of mathematical transformations involves understanding how the concepts encountered there fit into larger mathematical and cultural structures.

In spite of the substantial resources required, this review suggests further benefit to embedded classroom research for researchers who employ more integrated methods, design from a constructivist stance, and understand that students have the ability to construct their own mathematical ideas as opposed to merely “mirror[ing] those embodied in external instructional representations" (Gravemeijer & Stephan, 2002, p. 145). This philosophy respects the abilities of learners and "implies that students develop a high level of intellectual autonomy" (p. 147). Student autonomy allows researchers to observe learners as they engage with ideas through multiple entry points, extend their learning as needed, and connect with big ideas in mathematics. This approach also honors differing cultural approaches to mathematics learning, which may exist beyond the traditional canon.

Conclusion

Given the critical importance of making mathematical ideas accessible and meaningful for all students, we suggest that research on geometric transformations and symmetry should take an elevated position in the research agenda. This research might expand our thinking on how these topic areas can offer scenarios linked to the real-world and to students' lived experiences and how they can support culturally responsive pedagogies. Such research could broaden our perspectives on learning trajectories and encourage the incorporation of innovative technologies—for example, digital and computational environments, and games for learning. In terms of students' understandings, the complex relationships between motion and mapping understandings in geometric cognition highlight the need for future research to adopt diverse perspectives and methodologies to enrich our understanding and support inclusive classrooms spanning diverse cultures and epistemologies.

At the same time, our review also highlights the reliance of many teachers on learning trajectories / learning progressions as scaffolds for teaching. Teachers, particularly pre-service mathematics teachers, face substantial challenges in designing and developing culturally responsive teaching strategies that provide access to big mathematical ideas. Our findings further emphasize the need to assist teachers with assessment in integrated teaching approaches because traditional learning trajectories / learning progressions often do not provide clear guidance on assessing learning from more integrated perspectives.
Fundamentally, this review underscores the predominance of research focusing on individual cognition. While studies considering social and semiotic approaches *have* increased over the past decade, the review identifies the potential value of a greater paradigm shift towards more holistic understandings of learning environments. The review also highlights opportunities to leverage programmable environments and computational thinking, signaling the potential value in exploring how to harness synergies across technologies and media.

Research traditions on symmetry and geometric transformations have long emphasized the formality of congruence and geometrical proofs, fostering the development of strong mathematical skills aligned with curricular objectives. Nevertheless, considering a motion-to-mapping view of the plane as advocated by Klein's Erlangen program (Edwards, 2009) provides a scenario to cultivate intuitions as grounded in everyday experiences as the foundation for learners to construct a mapping view. This approach is naturally supported by discrete visualizations and animations of motion made possible by computational environments. Consequently, there is an open and intriguing venue for future research in exploring how virtual discrete spaces may facilitate (as well as how they may hinder) learning processes.

Increased research-practice partnership around classroom learning from an integrated approach could also give agency to students and teachers. Traditionally, dynamic geometry environments have been utilized to constrain students to manipulate systems according to pre-existing mathematical rules, inadvertently foreclosing the opportunity for them to engage in the development of geometric thinking actively (Yerushalmy & Houde, 1986). In this case, the technology becomes an outside authority, as is often also the case with learning trajectories. When authority for instructional activity and decision making is ceded to impersonal and external entities, teachers may feel they cannot deviate from the stated order of learning outcomes and thus become limited in their ability to envision creative ways of exploring mathematics with their students.

Finally, expanding the research on transformations to include under-researched ideas both *inside* mathematics education (e.g., spatial reasoning and equity and access) and *outside* the traditional mathematics research repertoire (e.g., computational thinking, Indigenous perspectives, and disciplinary practices in STEM) may "expand the possible" (Davis & Francis, 2023) so that learning about transformations broadens our thinking about mathematical learning in general to better serve all students.

Statements

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