



www.ijemst.net

The Story of Definite Integrals: A Calculus Textbook Narrative Analysis

Amanda Huffman Hayes 
Purdue University, United States

To cite this article:

Huffman Hayes, A. (2024). The story of definite integrals: A calculus textbook narrative analysis. *International Journal of Education in Mathematics, Science, and Technology (IJEMST)*, 12(1), 139-177. <https://doi.org/10.46328/ijemst.3045>

The International Journal of Education in Mathematics, Science, and Technology (IJEMST) is a peer-reviewed scholarly online journal. This article may be used for research, teaching, and private study purposes. Authors alone are responsible for the contents of their articles. The journal owns the copyright of the articles. The publisher shall not be liable for any loss, actions, claims, proceedings, demand, or costs or damages whatsoever or howsoever caused arising directly or indirectly in connection with or arising out of the use of the research material. All authors are requested to disclose any actual or potential conflict of interest including any financial, personal or other relationships with other people or organizations regarding the submitted work.



This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License.

The Story of Definite Integrals: A Calculus Textbook Narrative Analysis

Amanda Huffman Hayes

Article Info

Article History

Received:

10 November 2022

Accepted:

25 August 2023

Keywords

Narrative framework

Definite integral

Written curriculum

Calculus

Abstract

There is a plethora of textbooks to choose from. While most textbooks contain the same content, teachers need to understand the textbook they have chosen to support their students' learning, as textbooks can significantly impact student learning and teacher instruction. When searching for a way to understand the various textbooks, my research question became, How do calculus textbooks tell the story of definite integrals? My research is in response to the never-ending search for how teachers can select and use their resources. Also, this study contributes to understanding the relationships between stories and knowledge construction using Dietiker's (2015) narrative framework to analyze the written calculus curriculum. I wrote the specific stories of introducing the definite integral as told by five calculus textbooks. Analyzing how the stories developed disclosed the similarities and differences among the texts in teaching definite integrals. The analysis also revealed a variety of avenues to introduce and teach definite integrals, including a focus on the area under the curve, Riemann sums, or the Fundamental Theorem of Calculus, and some implications to the context of physics related to these avenues. These insights provide teachers with a deeper understanding of textbook lessons and the variances that can potentially alter student learning.

Introduction

Walking into a first-year calculus course at the secondary or collegiate level, one will likely find the same core components introduced, studied, and practiced. Students learn a skill and then see its application throughout a calculus course. Students learn about limits and then use limits to learn about derivatives. They then discover the purpose and application of derivatives before moving on to integrals, where the students learn the rules and then the application of integrals. After students learn calculus, some continue to other courses, specifically calculus-based physics, where they apply their mathematical knowledge of integrals. While there are differences from one classroom to the next based on the needs of the teacher and the students, do teachers who have a written curriculum start with a similar structure, a similar written curriculum?

Reflecting on my first-year teaching calculus at the secondary level, I relied heavily on the textbook the school provided, along with the help of a colleague who had taught the course for a couple of years. As with many first-

year teachers, my goal was survival; the textbook was my lifeline. It became the foundation of my instruction. Grossman and Thompson (2008) found other novice teachers in similar situations with the textbook becoming their foundation. The textbooks and curricular resources they used "seemed to have a profound effect on how they thought about and taught the subject matter" (p. 2019). Grossman and Thompson's finding certainly held for me as a novice calculus teacher. While my enacted curriculum, how I delivered instruction and interacted with the curriculum with students, altered as a result of professional development and trial and error in teaching calculus for several years, the initial interaction with the written curriculum continued to impact my enactment. My instruction might have been different if I had a different calculus textbook.

The teacher-curriculum relationship, as described by Remillard (2005), is multifaceted. Remillard (2005) noted that teachers' knowledge, beliefs, and dispositions" (p. 239) impacted the process of using mathematics curriculum, and therefore uniform mathematics instruction was not common. Therefore, teachers must have support and opportunities to study and learn new curricula, as the written curriculum shapes instruction, and instruction impacts student learning and achievement. Westwood Taylor (2016) noted that continued research on mathematics curriculum, specifically on teacher use, was essential to support teachers and curriculum developers. Furthermore, as standards, education, and curriculum use continue to evolve and change, there is a constant need for additional research to build on previous findings. Perhaps collectively, multiple research findings could lead to determining critical components of an effective mathematics curriculum, but finding the *best* will remain a challenge. Thus, I will examine the definite integral in the written curriculum of calculus while determining if different structures and sequences exist within the written curriculum for teaching calculus.

Schoenfeld (2001) argued that there are two main intertwined purposes for research in mathematics education: pure mathematics education research aimed at understanding "mathematical thinking, teaching, and learning" (p. 222) and applied mathematics education research aimed at "understandings to improve mathematics instruction" (p. 222). My research will primarily focus on pure mathematics education research, which can and will affect applied mathematics education research. My analysis will contribute to the never-ending search for "understanding the best ways in which a teacher can use his or her curriculum resources" (Westwood Taylor, 2016, p. 448).

Additionally, Larsen et al. (2017) noted a need for continued research in pure mathematics education. In 2013, in the United States alone, 28% of the postsecondary Calculus I students received a D, F, or "withdraw" from the course. Calculus I was often an academic roadblock, and there has been a push for students to take Calculus I at the secondary level in hopes of decreasing failures at the postsecondary level. Therefore, to improve student learning and achievement, teachers need to know and understand their curriculum.

By conducting this study, I am responding to the call to "develop a better understanding of how students negotiate the curriculum, where the pedagogical obstacles lie, and what can be done to improve student success" (p. 526). Students' success in calculus is not merely about passing the course. Oehrtman and Simmons (2023) stated "integration serves as perhaps the most prominent source of real-world applications of calculus content" (p. 40). Schoenfeld (2001) would call pure mathematics education research, focusing on integration, one of the areas of

least notability in the research of limits, derivatives, and integrals. Continued research on pure mathematics education will support research on applied mathematics education and improvement in student learning. Also, I am responding to the call for research on the relationships between stories, perhaps different types, and knowledge construction (Healy & Sinclair, 2007) while utilizing a narrative framework adapted from Dietiker (2015) to analyze the written calculus curriculum. In this study, I seek to answer the question, How do calculus textbooks tell the story of definite integrals?

Literature Review

In this section, I begin by summarizing current research on the use of curriculum by teachers and, therefore, the need for teachers to make sense of their curriculum. I then focus on the written calculus curricula and, more specifically, integration in the written calculus curricula. In particular, the approaches and sequencing of integration from a historical context and those found in written curricula are highlighted, along with the necessity for and ability of students to make meaning of the concepts of differentials and integrals and to transfer their calculus learning to other fields, particularly physics. Then, I expand on the aesthetic dimensions of mathematics and how the reveal of mathematical concepts, which creates the story of the lesson, can alter the concept. Finally, I discuss how a narrative framework writes the story and paints the picture of the reveal of the mathematical concepts, and helps readers make sense of the written curricula.

Teacher Use of Curriculum

There are numerous mathematics textbooks to choose from at all levels, and curriculum resources are representative of the authors' interpretation of how to construct the curriculum, sometimes based on standards and other times based on content perceived as significant. These authors transformed knowledge into learning opportunities for teachers and students (van den Ham & Heinze, 2018). As each author transformed the knowledge, differences emerged in the textbooks. Remillard et al. (2014) noted that textbooks "play a critical role in most school systems in communicating curricular expectations to teachers, transforming curricular goals into instructional plans, and supporting teachers to enact these plans" (p. 1). Since textbooks play the role of mediator between the teacher and students and thus impact instruction within a classroom, instruction in classrooms may differ depending on the textbooks utilized. In some classrooms, textbooks dominated teaching practices and dictated what students learned. van den Ham and Heinze (2018) found a variation in the curricular interpretation of the textbook, as well as the educational, cultural, and pedagogical intentions, which therefore had the potential to influence classroom instruction. Additionally, "according to the Trends in International Mathematics and Science Study (TIMSS) 2011 on average about 75% of the primary school teachers base their instruction on the mathematics textbook" (p. 134). These classrooms where teaching practices directly reflected the textbook contribute to the necessity for continued research and analysis of mathematics textbooks.

The Calculus Written Curriculum

According to Bressoud (2011), "calculus emerged because the geometric and dynamic conceptions of the integral

and derivative came to be seen as manifestations of common general principles, but it took time and genius to extract those general principles” (p. 100). There are two primary views of integration: the geometric conception based primarily on Leibniz’ description of integrals as area and the dynamic conception based on integration as “the accumulation of a quantity described by its rate of change” (p. 100) and favored by Newton and his views of integration. However, neither Leibniz nor Newton were the first to arrive at the geometric or dynamic conception, or the ideas of the Fundamental Theorem of Calculus (FTC).

Bressoud’s (2011) reflections on teaching the FTC credited many other mathematicians for their contribution to geometric and dynamic conceptions before Leibniz and Newton, including Barrow, Struik, Leahy, and Gregory. Gregory was perhaps the first to show “how to find the length of a curve by finding the area under a related curve” (p. 102). However, van Heureat’s work was the first published discovery of “the general formula for calculating arc length in terms of area under a related curve” (p. 102). Additionally, Barrow taught Newton at Cambridge University and made the first full discovery of the differentiation and integration inverse relationship. Moreover, “lying behind [Barrow’s] geometric argument [accumulator function] is a dynamic understanding, with the curve below the horizontal axis understood as the velocity and the curve above as the accumulated distance” (p. 102). However, Barrow, Struik, Leahy, Gregory, and van Heureat receive little credit, as it was Leibniz and Newton who studied the works of these and other mathematicians and connected the techniques to the application in the geometric conception with the hidden dynamic conception.

Furthermore, Boyer (1959) noted that similar ideas existed from Newton's predecessors, from Anitphon to Pascal, but Newton’s calculations were slightly different. If not for Leibniz and Newton’s predecessors from India, Egypt, and throughout Europe, Leibniz and Newton would not have had prior work to study and build on. Thus, Bressoud (2011) noted variations in the presentation of the FTC, or FTIC, the Fundamental Theorem of Integral Calculus, as others referred to it, exist, as mathematicians have compiled numerous findings to lead to the main results of the FTC and the geometric and dynamic significance in the study of calculus. Understanding geometric and dynamic significances can be difficult. Bressoud (2011) noted that students often struggle to shift back and forth between the two, which is how most textbooks are structured. However, if we follow historical order, calculus textbooks would begin with “the integral as the limit of sums of areas of rectangles” (p. 296) rather than differentiation. However, most textbooks begin with differentiation before any mention of integration, since differentiation is generally more conceptually understood than integration. Therefore, differentiation is before integration in calculus textbooks and is generally taught before integration.

Most syllabi for teaching first semester calculus included the following topics in order: (a) limits and continuity, (b) differentiation rules, (c) meaning of the derivative, (d) antidifferentiation, (e) the definite integral and the FTC, (f) trigonometric functions, (g) logarithmic and exponential functions (Steen, 1989). Bressoud (2011) found that when introducing integration, many contemporary textbooks began with Riemann’s definition by finding the summation of rectangles, yet most mathematicians of the seventeenth and eighteenth centuries who were doing calculus understood integration as an area, but worked with it as an antiderivative. He wondered why textbooks began with the area understanding and did not jump straight into the work with the antiderivative. My study of five calculus textbooks will reveal if this trend continues.

Integration in the Written Calculus Curriculum

Bressoud (2011) explored the origins of the FTC and how it became formalized in calculus textbooks in the nineteenth and twentieth centuries. He found that definite integrals initially appeared in textbooks in several ways. In 1820, Poisson introduced the traditional approach in which the difference of the antiderivatives is equal to the integral: $\int_a^b f(x)dx = F(b) - F(a)$, where $F(b) - F(a)$ is the definite integral. Leibniz had previously presented integrals in this way, but was unable to generate the mathematical formulation. Most American textbooks used the Poisson method. Bressoud and others argued for students to develop both conceptions of integration in order to understand its dual-nature. The students' understanding of the definite integral goes beyond simplicity; "the definite integral is the difference of the values of 'the' antiderivative" (p. 99). This simplistic understanding would result in little meaning of the FTC. Thus, students must also develop an understanding of the definite limit to have a more thorough understanding of integrals.

Sonfronas et al. (2011), in a study focused on key understandings of first-year calculus students, collected data from audio interviews with 24 participants, each of whom had a Ph.D. in mathematics, found the following components of integration essential for students' understanding of calculus: "(a) the integral as net change or accumulated total change, (b) the integral as an area, (c) techniques of integration" (p. 138). Similarly, Sealey (2006) noted that simply understanding the procedures for finding a definite integral was not sufficient, as several real-world scenarios involved functions where using the FTC was not possible. Therefore, students needed to understand definite integrals and know other ways to evaluate definite integrals, including Riemann sums. Additionally, an understanding of Riemann sums generally lent itself to understanding how to set up an integrand. Students often were unable to explain what integration meant, specifically in relation to area, and could simply perform the procedures (Larsen et al., 2017; Orton, 1983; Rasslan & Tall, 2002). As mathematics teachers, we need to find ways for students to understand more than just the procedures when working with integration; perhaps the sequencing of the written curriculum has been partly at fault?

In several research studies that focused on different approaches to integration and the definite integral, the key concepts and objectives of student learning varied. Thompson (1994) suggested the need for "image-building with accumulation, rate of change, and rate of accumulation" to help students grasp the FTC. Similarly, Sealey (2006) emphasized that a solid understanding of Riemann sums provided students with meaning and understanding when evaluating definite integrals and applying definite integrals. She argued that "students need to do something with the components of definite integrals in a way that reflects and is regulated by the underlying structure in order to be able to reflectively abstract and understand that structure" (p. 232). Her research suggested that Riemann sums provided a means for students to reflect on the underlying structure of definite integrals. Meanwhile, Orton's (1983) findings revealed that students could perform procedures, the objective of many teachers in his study, to find definite integrals, but "the students rarely could explain their procedures" (Sealy, 2006 p. 47). Students needed illustrations along with explanations to help solidify their understanding. Thus, Tall recognized the need for additional research into the approaches to teaching mathematics, as apparent in his comment (as cited in Gravemeijer & Doorman, 1999):

Mathematicians tend to make a typical error when they design an instructional sequence for calculus.

The general approach of a mathematician is to try to simplify a complex mathematical topic, by breaking it up into smaller parts, [which] can be ordered in a sequence that is logical from a mathematical point of view. (p. 112)

This error Tall found was similar to Orton's (1983) findings, since the students knew the procedures but not the meaning behind what they were doing. Therefore, in my analysis of the calculus textbooks, I will clarify the sequences of the authors, which will reveal geometric relationships, dynamic relationships, or both relationships.

Significance of Integration

Building students' understanding and ability to transfer their mathematical knowledge to other STEM fields, particularly physics, is essential in learning integration and calculus. For instance, integration is necessary in introductory calculus-based physics courses to solve problems, but often students can merely compute integrals and not apply them due to a lack of conceptual understanding (Amos & Heckler, 2015; Hu & Rebello, 2013; Jones et al., 2016). In order to be prepared to extend their mathematical knowledge to applications in physics among other fields, Amos and Heckler (2015) suggested students be equipped with and make use of "expert-like language and descriptions of differentials and differential products (e.g., Fdx or vdt)" (p. 35). "Fundamentally, the meaning of an integral can be understood as an infinite sum of differential quantities" (Amos & Heckler, 2015, p. 35). Other research echoes the importance of this understanding of accumulating infinitesimally small elements (Ely, 2020; Jones, 2015; Oehrtman & Simmons, 2023; Pina & Loverude, 2019).

In science and engineering courses, students need to be able to use their mathematical understanding. Specifically, students need to understand and be able to apply integration in calculus-based physics courses where students blend their mathematical understanding of concepts and notation with the physical world (Hu & Rebello, 2013). Students' transfer of their mathematical knowledge to physics applications does happen without challenges as students often struggle to appropriately apply concepts of integration in physics contexts (Nguyen & Rebello, 2011). Student recognized the need for integration but struggled in setting up and computing the desired integral: "incorrect expression for the infinitesimal quantity and/or accumulating the infinitesimal quantities in an inappropriate manner" (p. 10). Pina and Loverude's (2019) findings mirrored this difficulty for students to reinterpret their calculus knowledge in physical contexts. For instance, they stated, "in introductory electricity and magnetism for example, there are many instances in which there is no area to speak of when using integration to solve a problem. Being able to articulate the integral as a sum in these problems is not only helpful but a necessary step to success" (p. 450). Thus, attention to students' understanding and application of integration has increased in the last two decades.

Integration is a significant foundational concept for further mathematics used regularly in physical world phenomena. For calculus instructors, it is essential to understand that first-year calculus courses often have students planning to pursue science or engineering degrees and are more interested in applying calculus in their fields than pure mathematics. However, the science mathematics serves is often disconnected in the pure mathematics courses (Jones, 2015). Pina and Loverude (2019) claimed that "students leave their calculus courses

with a correct but incomplete understanding of the meaning of a definite integral (p. 447). Students need to be able to interpret the integral as a sum when applying integration in physical contexts (Pina & Loverude, 2019).

Jones (2015) examined three of the most common conceptualizations of the definite integral (i.e., (a) as the area under the curve, (b) as the values of an anti-derivative, and (c) as the limit of Riemann sums) often found in calculus textbooks. Specifically, he focused on the usefulness for “making sense, or having the general idea of how that concept applies to or relates to a given situation,” (p. 12) of integrals given the conceptualization in (a) a pure mathematics context, (b) an applied physics context, and (c) the overlap or disconnect between pure mathematics and applied physics. Jones used the lens of symbolic forms, or the arrangement of symbols in an equation of the definite integral (Sherin, 2001), to determine the usefulness. Sherin’s (2001) symbolic forms provide an avenue to survey the creation, interpretation, and cognition with quantitative relationships through equations. Jones (2015) found that “while all the symbolic forms showed their usefulness for decontextualized integrals, there was a significant difference in how productive certain understandings of the integral were when considering contextualized integrals” (p. 15). Oehrtman and Simmons (2023) suggested that to help students understand the differential as a measurable quantity, frame early instruction in physical contexts with varying quantities.

Aesthetic Dimensions of Mathematics

Mathematicians from Greece, Egypt, India, the Middle East, and western Europe made significant contributions to the mathematical field through their findings that incorporated beauty and aesthetics into mathematics. For example, Archimedes’ writings built on suspense and surprise by revealing his mathematical findings. The writings of Hellenistic mathematicians made the invisible visible. Variation, surprise, and paradox created beauty in their work (Netz, 2010). These mathematicians, while focused on the doing of mathematics, took time to develop the aesthetic of knowledge as it affected the reader’s understanding and disposition toward mathematics (Netz, 2010; Sinclair, 2004).

Mathematicians presented their original work with beauty; perhaps current mathematics textbooks also exhibit some of this same aesthetics. Similar to the significance of the way in which a story unfolds; methods of revealing or teaching mathematics are significant. The reveal and the method of the reveal of mathematics affect the storyline and the learning experience. Dietiker et al. (2016) focused on the aesthetic nature of mathematics teaching and learning through their research on how different enactments of the same written curriculum differ and the role of the differences in the aesthetic reactions of students. In this study, I investigate the aesthetics of the definite integral lessons of five textbooks by analyzing the variations, surprises, paradoxes, and other elements within the lesson. The findings will help answer my research question, How do calculus textbooks tell the story of definite integrals?

A Narrative Framework for Mathematics Curriculum

With continued curriculum reforms, teachers need ways to make sense of the ever-changing written curricula.

Teachers must understand all components of the curriculum where a lack of understanding can lead to varying enacted curricula. While varying enacted curricula are appropriate to address student needs, the written curriculum still influences the enacted curricula. Thus, teachers, including pre-service teachers, must learn how to understand and analyze the written curriculum provided to them. Sinclair (2005) suggested that teachers and students read mathematics texts as stories. Using the proof that the $\sqrt{2}$ is irrational, she, first allowed the reader to see the plain text with the appropriate mathematical notation to prove the irrationality of the $\sqrt{2}$. Then she invited the readers to listen as she added “the pace, motive, and human texture that transmute the haphazard chorus of events into drama, into story” (p. 7). From her study, she concluded that if teachers had the time to do this throughout the entire text, perhaps student engagement would increase as students came to see mathematics as more than just symbols and notation on paper.

However, unfolding the text into a drama-filled story is time consuming and would require teaching students how to do so as well if they were to read the text. Dietiker (2015) suggested that mathematics textbooks are stories and that these stories do not need any personal flair for the drama to unfold. Bal’s (2009) literary theory heavily influenced Dietiker’s (2015) narrative framework for a mathematical story. Bal proposed three interrelated layers, *text*, *fabula*, and *story*, through which to analyze the meaning of narratives. The *text* was the medium of the narrative, i.e., film, novel, live play. The changes in the narrative occurred when the *fabula*, or logical truths, were separated from the *story*, or revelations of the truth (Bal, 2009). Furthermore, the sequencing of the revelations impacted the story. For example, if all of the revelations occurred at the end of the story or lesson, then prolonged suspense occurred throughout the story. However, if the story slowly revealed answers or findings, then the reader experienced less suspense and possible closure throughout the story at various points within the story.

A mathematical story, as defined by Dietiker (2015), is “the interpretation of the chronological sequence of mathematical changes in a mathematics textbook by a reader” (p. 288). A mathematical story provides a context for logical mathematical events to unfold in a particular sequence while connecting a beginning and an end (Dietiker, 2015). The mathematical story was one of the three layers identified by Dietiker (2015) in relation to the three interrelated layers of a narrative by Bal (2009). Dietiker (2015) stated, “these layers recognize the roles of media (‘mathematical text’), content and meanings developed throughout the narrative (‘mathematical fabula’), and sequence and effects of the unfolding revelations (‘mathematical story’) of the mathematical sequences” (p. 288). The mathematical story depended on the sequencing, and a slight change in the sequencing could change the elements of suspense, drama, or surprise.

For example, in Thomas’ *Calculus: Early Transcendentals* (Weir et al., 2008) there is a two-part question and depending on which part the authors ask first, the story changes:

52. A. Suppose that the inequalities $\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$ hold for values of x close to zero. (They do, as you will see in Section 11.9.) What, if anything, does this tell you about $\frac{1 - \cos x}{x^2}$? Give reasons for your answer.

B. Graph the equations $y = \left(\frac{1}{2}\right) - \left(\frac{x^2}{24}\right)$, $y = (1 - \cos x)/x^2$, and $y = \frac{1}{2}$ together for $-2 \leq x \leq 2$. Comment on the behavior of the graphs as $x \rightarrow 0$. (p. 84)

In Part A, the authors presented an inequality that sandwiched the function of interest between two functions. The heading, Using the Sandwich Theorem, prompted the reader to evaluate the limit by using the outer two functions in the sandwich, thus utilizing the Sandwich Theorem. Moving to Part B, the reader already had an idea of what the graphs of the three functions should look like for values of x close to zero based on Part A. Therefore, the behavior of the graphs as $x \rightarrow 0$ would not be unexpected.

If the authors reversed the two parts and first asked the reader to graph the three functions, one may find the behavior of the graphs as $x \rightarrow 0$ to be surprising. Perhaps one would also comment that the graph of the function $y = (1 - \cos x)/x^2$ was between the other two functions, or sandwiched. Again, due to the heading Using the Sandwich Theorem, the text perhaps tipped the reader off that the two functions sandwich the third function. However, an element of surprise for the reader when looking at the graphs may still exist. Then the graphical representation helps the reader understand the given information, including the inequality, and allows the reader to further comment on the $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ instead of taking the text's guidance that the inequality statement involving the three functions was, in fact, true. This example illustrates that identifying and analyzing the sequence of elements of a text can provide insight into the connections throughout the story or the disconnections within the story. The narrative lens provided this insight and the potential impact on the learned curriculum as a result.

Additionally, the mathematical story can consist of a single lesson, such as continuity; multiple lessons, such as interpreting the graphs of f , f' , and f'' ; a unit, such as derivative rules, multiple units, or even the entire course. The various components of the story change as the magnitude changes; yet, each element of the story will still be present. However, with stories of greater magnitude, sequencing and coherence of sequencing became even more impactful, as evidence supported that altering sequencing altered meaning and relationships (Dietiker, 2015). Thus, in examining the story of integrals in this paper, I will examine singular lessons. In the next section, I will describe the elements of the story from the scale of a single lesson.

Dietiker (2015) noted that within every story, you have characters, actions, settings, and a plot; a mathematical story was no different. Mathematical characters are the mathematical objects introduced and given characteristics (e.g., functions, an expression, an operation, an image on a graph). The sequencing of the story plays a role in character development and can set the context for future understanding. Similarly, sequencing can contribute to the significance of the character and the impact of the character on mathematical objects. Mathematical objects can become mathematical characters over the course of the story. This is an element of character development. Mathematical action “can be interpreted as a manipulation of a mathematical character, such as a number or shape, resulting in a mathematical change” (p. 295).

Mathematical actions can occur once or multiple times, and their recurrence may or may not advance a story. Examining the actions can indicate the significance of the action. Mathematical actions that advance the story can introduce elements of surprise and even introduce new mathematical characters. The mathematical setting “is the manifestation of the content; that is, the mathematical representation” (p. 296). It is within the setting, whether it is an expression, equation, graph, or real-world application, that mathematical characters appear and interact. Similarly, to how actions advance the story, changes in the setting also tend to lead to advances in the story, as

the setting can further develop characters. Changes in the setting may challenge the reader or provide new knowledge of a character or action.

Furthermore, Dietiker (2015) noted that the combination of objects, characters, setting, and actions leads to plot formation. A mathematical plot “describes the aesthetic response of a reader as he or she experiences a mathematical story, perceives its structure, and anticipates what is ahead” (p. 298). The mathematical plot can ignite curiosity in the reader, especially if throughout the story questions are introduced and answers revealed while reading the story.

Richman et al. (2018) described the mathematical plot using three characteristics: density, rhythm, and coherence. The density of the plot increased as the number of questions increased, which added tension to the reader. As the reader began to answer the questions, the tension and density decreased. The rhythm was the result of the questioning style. The introduction of questions and answering of questions impacted the density, and therefore also the rhythm. The rhythm contributed to the reader's surprise and suspense and drew the reader's attention to important events through changes in rhythm. Finally, coherence was the connectedness, or lack of connectedness, of the events and the ideas of the mathematical story. If there were no connections, the reader would not be able to make predictions and would be surprised. The ability to make predictions offered a sense of completeness and allowed the story to connect and flow. Coherence and the similarities or differences in coherence of the stories is the main characteristic that I will be looking for in the plot development of the mathematical story told through textbook lessons on definite integrals.

A narrative framework provides educators with conceptual tools to make sense of a written mathematics curriculum. This article will demonstrate the use of the framework to specifically describe the story of the definite integral as told by five calculus textbooks. The reader is the only one who can interpret the mathematical plot and all of the elements of the story, as with any piece of art. Therefore, this paper represents my interpretations of mathematical stories and how I see the story of definite integrals unfolding in five calculus textbooks.

While I compare multiple texts, the purpose of this paper remains to demonstrate how a written curriculum can be analyzed through a narrative lens toward studying written calculus curriculum to provide teachers and researchers with insight into how they can, too, read a text and make sense of it. The purpose is not to distinguish one curriculum as superior to the others, but rather to utilize the narrative framework to provide the story told by each textbook. In this paper and in looking at the coherence of mathematical stories in multiple textbooks, my aim is to answer my research question, How do calculus textbooks tell the story of definite integrals?

Methods

I used Dietiker's (2015) narrative framework to investigate and write the stories of definite integrals found in five calculus textbooks. The narrative framework allowed me to write the stories of the definite integrals in each textbook and revealed the similarities and differences between the stories. My interpretations of these particular stories offer a teacher's reading and understanding of the textbook. Using the following methods, as a researcher,

I tried to remain consistent in examining the written text as it was written and not to make inferences or interpretations based on other experiences in writing each of the five stories told in the textbooks. Through the use of the literary framework, my stories of definite integrals based on the five calculus textbooks shed light on the coherence of definite integrals within a lesson and identified the similarities and differences among the textbooks in their approach to teaching definite integrals.

Textbook and Lesson Selection

In the United States, textbook writers must meet various state standards and compete for a school district to select their textbook, as there are numerous options (Leland et al., 2015). When searching for textbooks to analyze, I started with the textbook I used in my own undergraduate calculus course. Then, when searching for textbooks covering a range of approaches, I sought calculus textbook recommendations from undergraduate mathematics professors. Using these recommendations, I found a variety of approaches within the five selected textbooks. Although the selected textbooks of this study could be used to teach calculus at the secondary level, I did not select any textbook intentionally designed for the Advanced Placement (AP) Calculus curriculum to avoid variances in the curricula designed with the intention of the textbook to prepare students for the AP exam rather than simply teaching calculus. Furthermore, textbooks tailored to a particular assessment, such as the AP exam, would probably be more similar to those intended for calculus instruction.

More specifically, the lesson on the definite integral was situated in a different context within the specific chapter of each of the textbooks and provided insight into the varied sequencing that exists in the written calculus curricula. I will describe this variation in the Findings section. The five calculus textbooks utilized in my research were: *Calculus: Single Variable*, 5th edition (Hughes-Hallet et al., 2009), *Calculus*, Third Edition (Dietiker et al., 2017), *Calculus & Its Application* (Goldstein et al., 2010), *Thomas' calculus: Early transcendentals Update*, 11th edition (Weir et al., 2008), and *Calculus*, 3rd edition (Rogawski & Adams, 2015). The lesson(s) analyzed from each textbook were the lessons which introduced the concept of the definite integral. I selected the definite integral as the mathematical object of the story, as the definite integral is imperative to a calculus course and contexts beyond mathematics. Additionally, there is no known narrative framework analysis on written calculus curricula, and definite integrals are a fundamental concept of calculus.

Reading Mathematical Stories

When reading the mathematical stories in the textbooks, I attempted to approach the lesson as a learner and not as a calculus teacher. However, since I have taught calculus for five years, this was almost impossible. In an effort to help position myself as a learner, I analyzed student editions of the textbooks and completed each problem using the content knowledge the textbook provided, while attempting to not use my background as a calculus teacher. In doing so, I attempted to restrict my interpretations to what the textbook had made known in solving the problems and in writing the story the textbook told. In order to facilitate my approach to the analysis and reading of the mathematical story as a learner and not a teacher, I continually referenced earlier lessons in the textbook to unpack what prior knowledge students would have at this point in the story. For example, Dietiker et

al. (2017) lesson begins with “The Return of Fredo and Frieda” and examines velocity and distance graphs.

To answer the questions asked, I returned to Lesson 1.5.1 “How do position and velocity relate?” as directed in the text, to understand the prior knowledge that students should have before continuing the lesson. Referencing prior lessons also made known prior content knowledge in consideration of the unit, or even the entire textbook, as one larger story. This aspect potentially contributed to the drama and suspense of the story. While reading only the lesson focused on definite integrals, one may see more suspense and drama, but if it was not the first reveal of the concept, then it may not create suspense within the definite integral lesson story. Accounting for the reveal of all concepts was critical to writing an accurate story of definite integrals, and it was important to refer back to previous lessons while analyzing the definite integral lesson.

Generating Mathematical Plots

To generate the mathematical plot of the story, I utilized three levels of coding. When I read each lesson line-by-line and completed all tasks within the lesson, I first divided the lesson/story into acts. Then, I reread the lesson line by line and sub-coded the acts into story arcs as the story advanced. Finally, upon coding the story arcs, where questions developed, I again reread each arc and sub-coded it into elements of proposal, explicit question, partial answer, and disclosure to write the story of the lesson. You will find a full detailed list of the sub-codes in a subsequent section.

Coding the Acts of the Story

Dietiker (2013b) noted that when writing the story, it is important to dissect the components of the lesson in order to examine the sequencing of the story. Specifically, following Dietiker’s narrative framework and methods, I decomposed each lesson into acts. Each act contained an individual focus. As soon as the story progressed or changed, a new act started. Acts can be a single statement or question, or a group of statements or questions depending on the content's focus.

The first act of each story was the title of the lesson in the textbook, as the title could trigger the reader to think in a certain way. Thus, some acts were simply a phrase or title, while others are longer. As mentioned above, while splitting the story into acts, I positioned myself as a learner as opposed to a teacher to understand the concepts in an attempt to remain as true to the story as possible. A new act began when a character, the action or setting of the character changed. Each task with a different focus created a different act. Acts even split paragraphs within the book if the focus shifted within the paragraph. In Figure 1, I present an example of the coding of the first four acts of Lesson 5.2 “The Definite Integral,” in Hughes-Hallet et al. (2009, p. 264-271).

Act 1 was the title, “The Definite Integral.” Then Act 2 had a discussion of previous experiences and a preview of what was ahead in this story. Act 3 focused on Sigma notation, which led to Act 4, which incorporated Sigma notation in finding the limit, which ultimately led to the definite integral. The lesson continued with 11 additional acts. The continued analysis of these acts and the sub-coding into story arcs is shared in a subsequent section.

Act 1 **The Definite Integral**

Act 2 In Section 5.1, we saw how the distance traveled can be approximated by a sum of areas of rectangles. We also saw how the approximation improves as the width of the rectangles gets smaller. In this section, we construct these sums for any function f , whether or not it represents a velocity.

Act 3 **Sigma Notation**

Suppose $f(t)$ is a continuous function for $a \leq t \leq b$. We divide the interval from a to b into n equal subdivisions, and we call the width of an individual subdivision Δt , so

$$\Delta t = \frac{b-a}{n}$$

Let $t_0, t_1, t_2, \dots, t_n$ be endpoints of the subdivisions. Both the left-hand and right-hand sums can be written more compactly using the *sigma*, or summation, notation. The symbol \sum is a capital sigma, or Greek letter “S.” We write

$$\text{Right-hand sum} = f(t_1)\Delta t + f(t_2)\Delta t + \dots + f(t_n)\Delta t = \sum_{i=1}^n f(t_i)\Delta t.$$

The \sum tells us to add terms of the form $f(t_i)\Delta t$. The “ $i=1$ ” at the base of the sigma sign tells us to start at $i=1$, and the “ n ” at the top tells us to stop at $i = n$.

In the left-hand sum we start at $i = 0$ and stop at $i = n - 1$, so we write

$$\text{Left-hand sum} = f(t_0)\Delta t + f(t_1)\Delta t + \dots + f(t_{n-1})\Delta t = \sum_{i=0}^{n-1} f(t_i)\Delta t.$$

Act 4 **Taking the Limit to Obtain the Definite Integral**

Now we take the limit of these sums as n goes to infinity. If f is continuous for $a \leq t \leq b$, the limits of the left- and right- hand sums exist and are equal. The *definite integral* is the limit of these sums. A formal definition of the definite integral is given in the online supplement to the text at www.wiley.com/college/hugheshallett.

Suppose f is continuous for $a \leq t \leq b$. The **definite integral** of f from a to b , written

$$\int_a^b f(t)dt,$$

is the limit of the left-hand or right-hand sums with n subdivisions of $a \leq t \leq b$ as n gets arbitrarily large. In other words,

$$\int_a^b f(t)dt = \lim_{n \rightarrow \infty} (\text{Left-hand sum}) = \lim_{n \rightarrow \infty} (\sum_{i=0}^{n-1} f(t_i)\Delta t)$$

and

$$\int_a^b f(t)dt = \lim_{n \rightarrow \infty} (\text{Right-hand sum}) = \lim_{n \rightarrow \infty} (\sum_{i=1}^n f(t_i)\Delta t)$$

Each of these sums is called a *Riemann sum*, f is called the *integrand*, and a and b are called the *limits of integration*.

The “ \int ” notation comes from an old-fashioned “S,” which stands for “sum” in the same way that \sum does. The “ dt ” in the integral comes from the factor t . Notice that the limits on the \sum symbol are 0 and $n-1$ for the left-hand sum, and 1 and n for the right-hand sum, whereas the limits of the \int sign are a and b .

Figure 1. “The Definite Integral” Lesson Coded Acts (Hughes-Hallet et al., 2009)

Sub-coding within and Across the Acts

After coding the acts of the story, or focal points of the lesson, I examined the progression of the acts for story arcs. A story arc is “the transition from asking to answering a question” (Dietiker, 2013a, p. 3). Story arcs can last for one act or multiple acts if the question raised remains unanswered. Also, unlike in the acts, story arcs can and likely will overlap as asking and answering questions continues within a lesson. To determine the story arcs, I modified the sub-codes in Dietiker’s (2013a) theoretical narrative framework and placed the sub-codes within a calculus curriculum. Dietiker (2013a) adapted her codes from the framework of Barthes (1974) to analyze the reader's inquiry (p. 18). The sub-codes I used are in Table 1.

Table 1. Sub-codes Used for the Story Arcs (Dietiker, 2013a)

Sub-code	Description	Example within the Story
Proposal (PR)	Suggestion of something to learn	Suggestion of a pattern which could lead to inquiry: the <i>approximation improves as...</i>
Explicit Formulation (EF)	Question raised by the textbook	A textbook asking “Calculate the left-hand and right-hand sums”
Implicit Formulation (IF)	Question raised by the reader	A textbook stating “Now to take the limit of these sums as n goes to infinity”
Promise (PM)	Acknowledgment that the question will eventually be answered	A textbook stating “we will see how numerical approximation works”
Snare (S)	Misleading information in regards to answering a question	A false statement
Equivocation (EQ)	Ambiguity possibly leading one in the wrong direction	A textbook showing the reader approximating area through rectangles or only working with velocity functions
Jamming (JM)	Perception that a question may be unanswerable	A textbook stating it may not be possible to approximate the area under the curve
Suspended Answer (SA)	Delaying answering the question even though the reader may have the answer	A textbook shifting from a discussion of a calculator finding definite integrals exactly and then going back to approximating by hand
Partial Answer (PA)	Progress made toward answering the question	A textbook mentioning a definite integral, but then a reader approximating by hand and only finding exact using a calculator at the time
Disclosure (DS)	Answer explicitly revealed	A textbook answering a question such as “The area between the x -axis and the parabola is 2.4.”

Analyzing Mathematical Plots

Once I wrote the mathematical plots, coded the plots into acts, and then sub-coded for the story arcs, I analyzed the similarities and differences across the five plots. How did the story arcs line up from one plot to the next? Did one plot contain more or less extended story arcs? Were all the story arcs similar in length, and was the density, or number of arcs, common across the plots? These questions were considered in the analysis of the five plots. I looked for similarities and differences in the plot structures. Furthermore, examining the plot structures allowed me to identify patterns across the five lessons. I also noted the coherence of each of the plots when analyzing the patterns. Additionally, I noted the strategies (i.e., Riemann sums, FTC, limits, area under the curve) of which integration was being introduced and explained and how the strategy connected to developing an understanding that researchers have found to be productive when applying integration to other contexts. This analysis led to findings related to my research question, *How do calculus textbooks tell the story of definite integrals?*

Findings

Writing the mathematical story of the definite integral provided an in-depth comparison of an individual lesson from five different calculus books. Although the concept, the definite integral, introduced, and taught were the same, the coding and story arcs indicated key similarities and differences in styles. Each of the five lessons involved proposals and implicit and explicit formulation of questions, delayed and partial responses, and disclosure. All five lessons had between 13 and 16 acts made up of 16 to 22 story arcs. Each lesson shared an overarching question: *What is the definite integral?* However, the makeup and progression of the acts and story arcs varied from lesson to lesson, as the reveal of the answer to the question differed from lesson to lesson, as seen in the style of questions and the span of the story arcs. These variations created the unique mathematical story of each lesson and have various potential connections to contexts beyond mathematics, specifically physics.

The Stories of the Definite Integral

Each lesson tells its own story. The following vignettes tell the individual and unique story of the five lessons analyzed from my perspective. Although the explicit questions are in the text and would be the same for everyone telling each of these stories, the implicit questions, promises, ponderings, and stages of the answer reveal may vary from researcher to researcher, as objectivity is challenging with varied backgrounds of readers. After teaching high school for nine years, five of which I was teaching calculus, I have background knowledge that others who have not taught calculus may not have. Merging stories from multiple readers can make the story more objective and is in consideration for a future study. Although I may have skewed stories from my personal experiences, particularly as a calculus teacher, my stories and findings still reveal important information about the curriculum and important considerations and implications for teachers and students. Each vignette recounts the story of the definite integral as told by the particular textbook. I also provide commentary throughout the story as part of the analysis of the stories. The places where I elected to comment within the vignettes occur where I found the flow of questions altered or the focus in relation to the definite integral shifted. The coding and sub-coding of the calculus lessons revealed the evolution of the definite integral throughout the lessons, as well as the consistencies

and inconsistencies in that evolution. In Tables 2, 3, 4, 5, and 6 found further in this section, with each vignette, the mathematical plot diagrams of each lesson are presented. In sequential order, the questions raised in the lesson are listed in the first column. Each shaded cell denotes the acts, labeled in the top row, where the questions were asked, pondered or answered, and the collection of shaded cells forms the mathematical story. In addition, the mathematical plot diagrams include the specific coding of the story arcs to describe the development of the question and reveal the answer indicated by the letters in the shaded boxes. Finally, the vignettes, along with the coding and sub-coding presented in the mathematical plot diagrams, reveal the mathematical story of the definite integral through the views of five different authors. The findings also support the notion that all written curricula are not the same and reinforce the importance of teachers to understand their curriculum and the story they tell.

Hughes-Hallet et al. (2009) “The Definite Integral” Mathematical Plot Diagram and Vignette

“The Definite Integral” lesson found in Chapter 5, “Key Concept: The Definite Integral,” in Calculus: Single Variable, 5th edition, follows section “5.1 How Do We Measure Distance Traveled?” and proceeds sections 5.3 and 5.4 titled “The Fundamental Theorem and Interpretations” and “Theorems about Definite Integrals,” respectively. Figure 2 depicts the mathematical plot diagram of “The Definite Integral” lesson. The diagram contains the story arcs numbered as they appear in the lesson and breaks down the questioning style and reveal of the answers of each story arc, or question. The collection of shaded cells forms the story of the lesson. The letters in the cells represent the following sub-codes: a: Proposal, b: Explicit Formulation, c: Implicit Formulation, d: Promise, e: Snare, f: Equivocation, g: Jamming, h: Suspended Answer, i: Partial Answer, J: Disclosure.

	Act	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1 What is the definite integral?																
2 Why would we construct sums for any function f?																
3 How would we construct sums for any function f?																
4 How small can the width of the rectangle become?																
5 How does taking the limit of the sums as n goes to infinity relate to integrals?																
6 What are these sums called Riemann sums?																
7 How are the limits of integration connected with n in the sigma notation?																
8 How can definite integrals be calculated by hand?																
9 How does numerical approximation work?																
10 What does it mean to calculate an integral to a desired accuracy?																
11 What is the left-hand and right-hand sums with n=2 of the integral from 1 to 2 of (1/t)dt?																
12 What is the relationship between the left- and right- sums for n=10 and n=250 and the integral?																
13 How does the integral from 1 to 2 of (1/t)dt equal ln(2)?																
14 How can we interpret each term if f(x) is negative?																
15 How can you interpret the integral as area?																
16 How can you find integral's exact value?																
17 What is the integral using a calculator or computer?																
18 How do you use a calculator to get this answer?																
19 What happens to the area under the curve if the graph doesn't lie above the x-axis?																
20 How does the definite integral relate to the area between the parabola and the x-axis?																
21 How did they get that answer in the calculator?																
22 How do you interpret the definite integral from 0 to sqrt(2) of sin(x^2)dx in terms of area?																

Figure 2. Mathematical Plot Diagram of Hughes-Hallet et al. (2009)

Figure 2, the mathematical plot diagram of the lesson “The Definite Integral,” shows a variety of lengths of story arcs creating its unique story. The overarching questions, or story arcs, of the story include constructing sums for any function and why we would even want to or need to do this in addition to using Riemann sums to help with the process. Additionally, another longer story arc beginning in Act 7 and spanning 6 acts involves answering the question *How can we interpret each term if f(x) is negative?*. The area under the curve in relation to the definite integral is a primary focus of this story, as indicated by the story-arc questions. The vignette of this lesson provides a more detailed glimpse and breakdown of the story than the mathematical plot diagram alone and discusses more about the connection between area and the definite integral in this story.

The lesson began with the first act, the title of the lesson, “The Definite Integral” (p. 264) (ACT1). This was the first mention of an *integral* in this text and, as such, raised the first question: *What is the definite integral?* (Q1). The main text of the lesson then began with reference to the previous section and recalling how we can approximate the distance traveled by the sum of the areas of rectangles and how the approximation improved with smaller and smaller widths of rectangles (ACT2). Act 2 concluded with a connection to the current section and inspired the question: *Why would we construct sums for any function f ?* (Q2). Additionally, the conclusion of Act 2 introduced another question: *How would we construct sums for any function f ?* (Q3). Questions 1, 2, and 3 remained unanswered as the lesson moved into “Sigma Notation” (ACT3). The discussion of sigma notation (ACT4) and the idea of the width of an individual rectangle inspired the following question: *How small can the width of the rectangle become?* (Q4). The text resolved the question in Act 4, as the lesson moved to obtaining the definite integral from taking the limit and specifically focused on “the limit of these sums as n goes to infinity”. Just as the text answered Question 4, the text promised Question 5: *How does taking the limit of the sums as n goes to infinity relate to integrals?* (Q5). Act 4 continued by introducing the notation of the integral and the making the connection to the sigma notation for the summation of the rectangles of a given width and the text partially answered Question 5. Additionally, the text included the terms “Riemann sum”, “integrand”, and “limits of integration” and the reader may wonder: *Why are these sums called Riemann sums?* (Q6). The notation and explanation of the notation raised another question: *How are the limits of integration connected with n in the sigma notation?* (Q7) before Act 4 was concluded.

The mathematical story began developing with the lesson title and quickly evolved with explanations of the notation for Riemann sums and integrals. Starting an explanation of the definite integral with Riemann’s sums is not surprising. Ely’s (2017) found nearly all calculus textbooks use Riemann sums to define the definite integral. However, Jones et al. (2016) cautioned that students’ perception of Riemann sums was that of a procedure that could be used to accurately estimate an integral, not a conceptual understanding, which was often a result of how it was taught. The text quickly invited the reader to question their connection and how the construction of these sums, particularly as n goes to infinity, is connected to integrals. At this moment in the story, the attention focused on summation and Riemann sums (Q2, Q3, Q4, and Q6) in lieu of the definite integral (Q1, Q5, and Q7). The critical component of Riemann sums bridged Lesson 5.1 and the sums of areas of rectangles into Lesson 5.2 and the eventual connection to the definite integral. The text left many unanswered questions as the story transitioned into Act 5.

The main text transitioned into “Computing the Definite Integral” (p. 265) and explained how technology could do this, but also how to do it numerically, which caused the reader to wonder: *How can definite integrals be calculated by hand?* (Q8). The text then named the numerical approximation method and the following explicitly proposed questions: *How does numerical approximation work?* (Q9) and *What does it mean to calculate an integral to a desired accuracy?* (Q10). The text started to reveal the answers to Questions 8, 9 and 10 in Example 1 (ACT6) while asking and answering more explicit questions: *What is the left-hand and right-hand sums with $n=2$ for $\int_1^2 \frac{1}{t} dt$?* (Q11) and *What is the relationship between the left- and right-hand sums of $n=10$ and $n=250$ and the integral in Question 11?* (Q12). The text revealed the answers using the left- and right-hand sum formulas provided in Act 3 in the discussion on sigma notation. When revealing the answer to Questions 11 and 12, the text

left the reader with another implicit question, as the text provided the exact solution, “ $\int_1^2 \frac{1}{t} dt = \ln 2 = 0.693147\dots$ ” (p. 266): How does $\int_1^2 \frac{1}{t} dt = \ln 2$? (Q13). No numerical or algebraic explanations were provided. The text revealed the solution visually on a graph and stated the answer. This left the reader to wonder.

In Act 6, the text explicitly posed Questions 11 and 12 and quickly provided answers to the questions in the same act, while also igniting an implicit question. The text stated “ $\int_1^2 \frac{1}{t} dt = \ln 2$.” This definite integral statement represented equivocation in this lesson, as the reader might have believed that the definite integral of all functions would result in a value expressed as the natural logarithm of a number. In examining prior knowledge, the text did not instruct the reader how to compute a definite integral algebraically, thus the cause of this question and the possible ambiguity in the computation of integrals, which can extend beyond calculus for students who go on to calculus-based physics or other sciences requiring the use of integration..

Act 7 began with another transition point in the lesson with a new subheading, “The Definite Integral as an Area” (p. 266). The main text of Act 7 began with the statement “if $f(x)$ is positive,” which inspired the question: How can we interpret each term if $f(x)$ is negative? (Q14). This question lingers, as the text only discusses a positive $f(x)$ in Act 7 and subsequent Acts. Acts 8 through 12 focused on the function $f(x) = \sqrt{1 - x^2}$ and considered the integral $\int_{-1}^1 \sqrt{1 - x^2} dx$. The text then explicitly asked: How can you interpret the integral as area? (Q15, ACT8), How can you find the integral’s exact value? (Q16, ACT9), and What is the integral using a calculator or computer? (Q17, ACT10). The text directly answered each of these questions immediately after asking the question. Then Act 11 interpreted the integral as the area of a semicircle with radius 1 and provided a figure with the function graphed, while Act 12 stated the answer from a calculator. Specifically, Act 12 stated that “a calculator gives the value of the integral as 1.5707963...” leaving the reader to wonder: How do you use a calculator to get this answer? (Q18).

Acts 7 to 12 revolved around the idea of a positive function and finding its integral by using the area under the curve and a calculator. Jones (2015) found the area under the curve model to be inadequate in helping students understand the quantity the integral calculates in the context of physics. Yes, the area under a curve was the idea of Lesson 5.1 and prior knowledge of the students would help answer the questions. However, Act 12 states and provides an approximated answer from a calculator but does not provide any guidance for calculator use. Therefore, the reader may wonder how to compute the integral with or even without a calculator at this point.

Act 13 shifted the focus of the lesson to “When $f(x)$ is Not Positive” (p. 267). Although the reader might be able to predict the answer from previous examples, the text presented the following question: What happens to the area under the curve if the graph does not lie above the x -axis? (Q19). The text explained the answer and reinforced it with an example asking the following question: How does the definite integral $\int_{-1}^1 (x^2 - 1) dx$ relate to the area between the parabola $y = x^2 - 1$ and the x -axis? (Q20). The text provided the answer from a calculator, as well as, an explanation incorporating a graph of the functions to show the relationship. However, once again, as a result of a calculator answer, the text left the reader to wonder: How did they get that answer in the calculator? (Q21). The text posed the final question: How do you interpret the definite integral $\int_0^{\sqrt{2}} \sin x^2 dx$ in terms of area? (Q22)

and answered it in Example 4 in the text (ACT14). The final Act, Act 15, generalized Riemann sums to conclude the lesson.

Acts 13 and 14 built on the connection of finding the area under the curve with both the area above and below the x -axis. Act 15 connected back to Question 7 from Act 4 in providing an attempt to answer more about Riemann sums. At the end of the story, there were a few questions that remained unanswered, including the following. How to use a calculator to find the definite integral? and How the definite integral of $\int_1^2 \frac{1}{t} dt = \ln 2$?

Dietiker et al. (2017) Section 4.1.1: “How can I calculate the exact area?” Mathematical Plot Diagram and Vignette

In Calculus, Third Edition, Section 4.1.1: “How can I calculate the exact area?” begins Chapter 4, “The Fundamental Theorem of Calculus.” Following Section 4.1.1, the chapter continues to discuss Riemann sums, the limits of integration, $+C$ in indefinite integrals, the FTC, application problems, and the area between two or even three curves. Figure 3 depicts the mathematical plot diagram of the “How can I calculate the exact area?” lesson. The diagram contains the story arcs numbered as they appear in the lesson and breaks down the questioning style and reveal of the answers of each story arc, or question. The collection of shaded cells forms the story of the lesson.

CPM	Act	1	2	3	4	5	6	7	8	9	10	11	12	13			
1 How can I calculate the exact area?		bi			d					ij							
2 What is a definite integral?			ci							ij							
3 How is Fredo's data reflected in Frieda's graph?				bi													
4 How is Frieda's data reflected in Fredo's graph?				bi													
5 What is the forward and backward connection between slope and area?				bi													
6 How can the exact slope of a curve at a point be found?					bi												
7 How can the exact area under a curve be determined?						bi				ij							
8 What are your observations from examining the four different graphs shown?							bi										
9 How can we calculate an exact area?								bi		ij							
10 What happens to delta x as more rectangles are used?								bi	i	ij							
11 How can you use a Riemann sum to write an expression to represent the exact area under f on [a,b]?									bi								
12 Will delta x ever equal 0? Why or why not?										bi	ij						
13 What happens to the area of each individual rectangle as n approach infinity?											bi	ij					
14 If the area is composed of rectangles with areas that are approaching zero, why does the overall area not approach zero?												bi	ij				
15 Is there a more efficient way of calculating a definite integral instead of using limits with Riemann sums?													ci				
16 How do you find definite integrals of functions with removable or jump discontinuities?														a			
17 What do the upper "b," and lower "a," bounds of the definite integral represent?														bi			
18 How are (b-a)/n in the limit and dx in the definite integral related?															bi		
19 Why is it important that we are multiplying (x^2 + 1) by dx?																bi	
20 What is the region on the represented by the definite integral and what is the area of the region?																	bi

Figure 3. Mathematical Plot Diagram of Dietiker et al. (2017)

The mathematical plot diagram of the lesson “How can I calculate the exact area?” in Figure 3, indicates two overarching story arcs, How can I calculate the exact area? and What is a definite integral?, which are eventually completed in Act 9. Many other story arcs, while not starting in Act 1 or 2, also wrap-up in Act 9. Act 9 appears to provide conclusions to many open questions, but also begins two new questions for the reader. The vignette of this lesson provides a more detailed glimpse and breakdown of the story than the mathematical plot diagram and reveals more about exactly what is happening in Act 9 in addition to the other acts.

The authors divided Section 4.1 into three subsections. The story began with the title of Section 4.1.1 (ACT1), which was also the first explicit question of the story: “How can I calculate the exact area?” (Q1). The focus changed in Act 2 with a subheading of “Definite Integrals” and a proposal and implicit question: What is a definite

integral? (Q2). Act 3 shifted to an application problem, as the text directed the reader to examine two velocity graphs and summarize the answers to the next two questions: How is Fredo's data reflected in Frieda's graph? (Q3) and How is Frieda's data reflected in Fredo's graph?" (Q4). Act 3 continued with a specific prompt related to the Fredo graph and the Frieda graph. The reader first visited the scenario with Fredo and Frieda in Chapter 1 and the text invited the reader to return to that section to recall and build on prior knowledge. Building on the prior understanding of these two graphs, the reader saw that velocity and distance were the focus of the two graphs. This led to the first prompt, and the text confirmed how the data of each student related to the other using either a derivative, where the text reminded the reader to think about the slope, or an integral, where the text reminded the reader to think about the area under the curve. The first prompt asked the explicit question: What is the forward and backward connection between slope and area? (Q5). The second prompt led to Act 4 with a shift in focus to secant lines and average velocity with the explicit question: "How can the exact slope of a curve at a point be found?" (Q6). The third prompt continued to focus on the same situation, which led the reader to Act 5 and another explicit question: How can the exact area under a curve be determined? (Q7).

Although the story began with an introduction of two new ideas from Questions 1 and 2, the story then shifted to building on prior knowledge and challenging the reader to make a conjecture through Acts 3, 4, and 5. The text invited the reader to think about the connection between slope and area under the curve, secant lines, and how an exact area under a curve could be found, linking directly back to Question 1 and to Question 2. These connections being made through the start of this story echo the need Jones (2015) found for students to be able to make connections in context of interpreting the definite integral as the area under the curve.

Act 6 continued with the overall theme of building on prior knowledge, as the text revisited Riemann sums and directed the reader to three more prompts. The three prompts were directly related and built on each other in order to develop similar ideas to be able to formulate an answer to the third prompt. The initial explicit question: What are your observations from examining the four different graphs shown? (Q8) invited the reader to make some observations, while the explicit follow-up question: "How can we calculate an exact area?" (Q9) tried to build on those initial observations. Then the final explicit question: "What happens to x as more rectangles are used?" (Q10) connected with the observations, and the students' responses to Question 9 could have contained the answer. Act 7 seamlessly flowed from the conclusion in Act 6, but challenged the reader to take the observations and use mathematical notation instead of simply words in working with Riemann sums. In Act 7, the guiding question was "How can you use a Riemann sum to write an expression to represent the exact area under f on $[a,b]$?" (Q11). Then in Act 8, the text presented three follow-up questions that refer to the reader's findings in Act 7: "Will Δx ever equal 0? Why or why not?" (Q12), "What happens to the area of each individual rectangle as $n \rightarrow \infty$?" (Q13), and "If the area is composed of rectangles with areas that are approaching zero, why does the overall area not approach zero?" (Q14).

Acts 6 and 7 built on each other and shifted the focus, or setting, of the story, to Riemann sums, the conception of definite integrals Ely (2017) found in nearly all calculus textbooks. Where does the definite integral fit within Riemann sums? The text provided the reader with four visuals to see how increasing the number of rectangles fits with approximating the area under the curve. These observations and questions in Act 6 transitioned the reader

into Act 7 and the development of a possible mathematical equation for Riemann sums. Then Act 8 challenged the reader to make observations about the equation developed and also allowed the reader to check and understand the equation in response to Question 11. The text revealed no answers at this point and provided no explanations outside of the question prompts other than a brief explanation about how Riemann sums could only make approximations.

The big reveal of the lesson took place in Act 9 when the text provided the reader with confirmation to several of the previous questions. There was a big shift in the story structure, as now the text informed the reader of information instead of asking questions for consideration. The text answered an overarching question from Act 2: What is a definite integral? (Q2), along with several of the other questions that involve Riemann sums. The text explained the connection between limits, Riemann sums, and the definite integral. These connections may be beneficial in helping students' "highlight the intrinsic worth of the Riemann sum as an important conceptual entity in and of itself" (Jones, 2015, p. 25), which not all textbooks do. Furthermore, the text provided an explanation and notation, including an explanation of the notation, of the definite integral. With the explanation, two implicit questions arose: Is there a more efficient method for calculating a definite integral than using limits and Riemann sums? (Q15) and How do you find definite integrals of functions with removable or jump discontinuities? (Q16). So, while there were answers provided, there were more questions to keep the reader thinking.

Act 9 gave the reader the long-awaited confirmation of the work done, as well as some connections to the questions asked. This is the first time in the story that the reader could relax a bit and absorb information instead of formulating answers and conclusions. However, Questions 15 and 16 continued to provoke the reader's thinking.

The next series of acts directly reflected the material presented in Act 9 on the definite integral and required the reader to unpack and make meaning with three distinct explicit questions relating to the formulation of the definite integral. Act 10 presented the question: "What do the upper 'b,' and lower 'a,' bounds of the definite integral represent?" (Q17). Act 11 focused on the $\frac{b-a}{n}$ in the limit in relation to dx in the definite integral. The final piece of the definite integral was to make sense of the multiplication of the function rule with dx (ACT12) and required the reader to explain why the operation was multiplication. Acts 10 - 12 provided a self-check for the reader to see if the reader absorbed and comprehended the information provided in Act 9. All three acts focused on a different part, each unique and equally important, of the structure of the definite integral through explicit questions. Once the structure was understood, the reader was ready to transition to the next lesson, where they would apply the concepts learned in this lesson. Also, perhaps the story of this lesson supports Nguyen and Rebello's (2011) four steps of integration in physics application questions, "(i) recognize the need for an integral, (ii) set up the expression of the infinitesimal elements, (iii) accumulate the infinitesimal elements, and (iv) compute the integral" (p. 3).

Goldstein et al. (2010) "Definite Integrals and the Fundamental Theorem" Mathematical Plot Diagram and Vignette

The "Definite Integrals and the Fundamental Theorem" lesson found in Chapter 6, "The Definite Integral," of

Calculus & Its Application follows sections titled “Antidifferentiation” and “Areas and Riemann Sums” and proceeds “Areas in the xy -Plane” and “Applications of the Definite Integral.” Figure 4 shows the mathematical plot diagram of “Definite Integrals and the Fundamental Theorem” lesson. The diagram contains the story arcs numbered as they appear in the lesson and breaks down the questioning style and reveal of the answers of each story arc, or question. The collection of shaded cells forms the story of the lesson.

	CIA													
	Act	1	2	3	4	5	6	7	8	9	10	11	12	13
1	What is a definite integral?	ci	i			i	i	i	i					i
2	What is the Fundamental Theorem of Calculus (FTC)?	ci					ci						ij	ij
3	How would we interpret the definite integral if the function was negative?		ci	h	ij									ij
4	What is the integral from 1 to 4 of $[(1/3)x+(2/3)]dx$?				bij									
5	What is the integral from 0 to 5 of $(2x-4)dx$?					bij								
6	Why is the FTC true?						chi							
7	What is the integral from 1 to 4 of $[(1/3)x+(2/3)]dx$ using FTC?						bij							
8	What is the integral from 0 to 5 of $(2x-4)dx$ using FTC?						bij							
9	What is the integral from 2 to 5 of $(3x^2)dx$?							bij						
10	What is the integral from 0 to 2 of $[(1/3)e^{(3x)}]dx$?							bij						
11	What is the area under the curve $y=x^2-4x+5$ from $x=-1$ to $x=3$?								bij					
12	How does the amount of change as the area under the graph align with the FTC?									ci				
13	What does the region representative of the distance the rocket traveled from $t=40$ to $t=100$ seconds look like?										bij		i	
14	What is the distance the rocket traveled from $t=40$ seconds to $t=100$ seconds?										bij		i	
15	What is the amount of oil consumed from 1972 to 1974?											bij		
16	How does the amount of oil consumed from 1972 to 1974 represent an area?											bij		
16	Why does the FTC work with definite integrals and area under the curve?												cij	ij

Figure 4. Mathematical Plot Diagram of Goldstein et al. (2010)

In Figure 4, the mathematical plot diagram of the “Definite Integrals and the FTC” lesson, it is apparent that the overarching story arcs are *What is a definite integral?*, *What is the FTC?*, and *How would we interpret the definite integral if the function was negative?*. These three story arcs span nearly all 13 acts as the answers are slowly revealed a little at a time within each act and as the other story arcs are developed and completed. The other story arcs span primarily one act and involve asking and answering the question all within that one act. This leaves little time for the reader to think through the answer on their own. The vignette of this lesson provides a more detailed glimpse and breakdown of the story than the mathematical plot diagram.

In lesson 6.3, the title “Definite Integrals and the Fundamental Theorem of Calculus” created Act 1. The title invited two implicit questions to start the lesson: *What is a definite integral?* (Q1) and *What is the FTC?* (Q2). Act 2 immediately provided an explanation for the definite integral in terms of Riemann sums, a topic discussed in the previous section. The text provided a quick explanation and revealed the notation in terms of the limit as $\Delta x \rightarrow 0$ with the Riemann sums calculation. The text clarified that “*the definite integral of a nonnegative function $f(x)$ equals the area under the graph of $f(x)$* ” (p. 316) (Q3). This explanation developed an implicit question of what if the function is negative. Act 3 progressed the story by inviting the reader to “calculate $\int_1^4 (\frac{1}{3}x + \frac{2}{3}) dx$ ” (p. 316) and then revealed the answer with an explanation for the definite integral using the concept of area under the curve. Immediately, Act 4 turned attention to answer Question 3 and addressed negative functions through a detailed explanation and a figure providing the visual representation of what the text described. The text generalized the combination of a negative and non-negative function over given intervals in Act 5 before the text presented the reader with the following example “calculate $\int_0^5 (2x - 4) dx$ ” (p. 317). This example allowed the reader to demonstrate their understanding of finding the definite integral for a function with non-negative and negative area.

The first five acts of the story set the stage for finding definite integrals using strategies previously studied,

including Riemann sums and the area under the curve. While the authors made connections between Riemann sums and area under the curve with the definite integral, it is unclear if this connection resonates with the students as researchers have found is needed (e.g., Jones (2015), Ely (2017)). Three specific examples to practice understanding the situations separately and combined allowed the reader to further develop an understanding of both nonnegative and negative functions.

A key piece of the plot developed in Act 6 to introduce the FTC for finding the definite integral opposed to using area under the curve or Riemann sums calculations previously discussed. However, an implicit question formed in response: *Why is the FTC true?* (Q6). Act 6 continued by inviting the reader to use the FTC to find two definite integrals in Questions 7 and 8. A detailed explanation and the answer followed each question. Act 7 invited the reader to continue to evaluate definite integrals, but did not specify the method in Questions 9 and 10. Again, a detailed explanation including the answer immediately followed these explicit questions. Act 8 transitioned with the explicit question to “compute the area under the curve $y = x^2 - 4x + 5$ from $x = -1$ to $x = 3$ ” (p. 319). Although in the previous four examples, the explanations mentioned using an antiderivative, this time the explanation included a mathematical notation to represent all work after acknowledging that $f(x)$ was nonnegative.

Acts 6 and 7 explored the FTC to find definite integrals. The text mentioned finding the antiderivative of the function before using the mathematical notation to find each definite integral. The same process continued for all four examples, finding definite integrals. The different structure in the question in Act 8 provided an opportunity for the reader to connect the task of finding the area under the curve and finding a definite integral, while it also invited the reader to set up the definite integral, a skill needed for extending one’s mathematical understanding into other contexts. In Act 8, all calculations were computed in the most efficient way possible.

In Act 9, the story shifted its focus to applications with definite integrals with the focus on area under a curve as “the amount of change” (p. 319), a previously developed concept. This prior knowledge connected directly to the FTC rewritten with taking the integral of a rate of change, or derivative function (ACT9). After an explanation and making connections, the text provided the reader with a scenario involving the position of a rocket and describing the area representative of the distance the rocket traveled over a given time interval in addition to finding the actual distance traveled. Act 11 presented a new, but similar, scenario involving oil consumption. Again, the reader had the task of finding the oil consumption over a period of time and then representing that as an area.

Acts 9 - 11 revealed the application of the definite integral and continued to reinforce the connection to area. The text reversed the tasks given to the reader in the rocket scenario (ACT10) and the oil consumption scenario (ACT11): describe the distance in relation to an area, then compute the distance versus finding the oil computation, and then relating it back to area. The different order in questioning reinforced the connection between the ideas.

Acts 12 and 13 concluded the lesson and the story. These two acts were similar and presented two cases to validate the FTC. Act 12 focused on the Riemann sum definition of the integral, while Act 13 focused on the relationship between area and antiderivatives only applied when $f(x)$ is non-negative. Acts 12 and 13 completed the story with the reasonings behind the main character, definite integrals, or more specifically, the FTC, throughout the story

in this lesson. Early in the lesson, the question implicitly asked why the FTC was true. The lesson ended with a reveal of the answer from two perspectives after the reader had the opportunity to practice using the FTC to find definite integrals.

Weir et al. (2008) “The Definite Integral” Mathematical Plot Diagram and Vignette

In Chapter 5, “Integration” of Thomas’ calculus: Early transcendentals Update, 11th edition, the reader finds the “Definite Integral” lesson sandwiched between “Estimating with Finite Sums” and “Sigma Notation and Limits of Finite Sums” and “The FTC,” “Indefinite Integrals and the Substitution Rules,” and “Substitution and Area Between Curves.” Figure 5 shows the mathematical plot diagram of “The Definite Integral” lesson. The diagram contains the story arcs numbered as they appear in the lesson and breaks down the questioning style and reveal of the answers of each story arc, or question. The collection of shaded cells forms the story of the lesson.

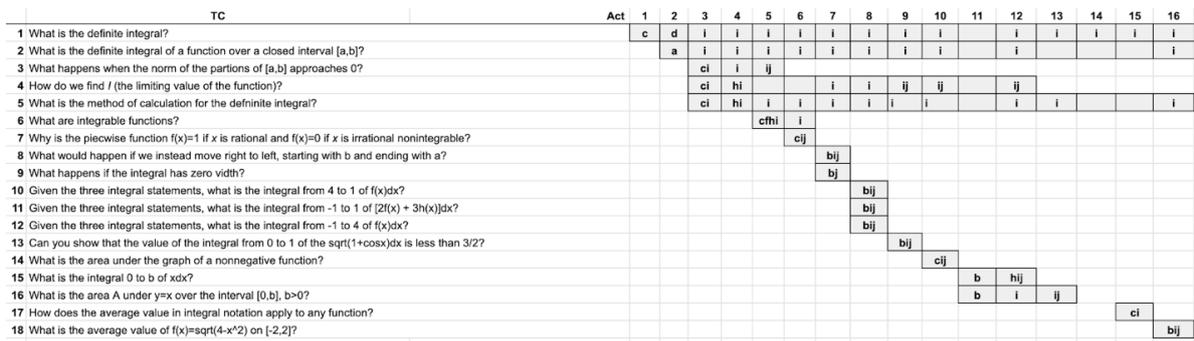


Figure 5. Mathematical Plot Diagram of Weir et al. (2008)

Looking at the mathematical plot diagram of the lesson “The Definite Integral” in Figure 5, the overarching story arcs, What is the definite integral? and What is the definite integral of a function over a closed interval [a,b]?, span all 14 acts as the answers are slowly revealed a little at a time within each act and as the other story arcs are developed and completed. There are two other story arcs, How do we find l (the limiting value of the function)? And What is the method of calculation for the definite integral?, that span multiple acts and build from additional questioning, or story arcs. The remaining story arcs are short and involve asking and answering the question within one act, leaving little time for the reader to think through the answer on their own. The vignette of this lesson provides a more detailed glimpse and breakdown of the story than the mathematical plot diagram.

“The Definite Integral,” (p. 370) the title of section 5.3 invited the first implicit question: What is the definite integral? (Q1). Act 2 recalled the study of Section 5.2 on the limit of finite sums of a function over an interval and connected prior knowledge with the focus of this section, finding the definition of the “definite integral of a function over a closed interval [a, b]” (p.370). The search for the definition of the definite integral would later provide an answer to Question 1. Act 3 progressed into building on and presenting the definition of the definite integral as a limit of Riemann sums.

The text developed the definite integral and worked towards its definition using limits as the main idea of the story of this lesson. Act 1 established the main idea of the lesson, and Acts 2 and 3 reinforced it. Questions for the

reader to process in the first three acts focused on what a definite integral was and finding that definition in relation to Riemann sums, common to nearly all calculus textbooks (Ely, 2017). However, the text quickly revealed the formal definition of the definite integral as a limit of Riemann sums in Act 3. Ely (2020) found studies show students are not developing a strong quantitatively-based understanding of calculus when taught calculus with limits. This has been shown to limit students' ability to interpret and model situations with calculus.

After revealing the definition of the definite integral in Act 3, the text established the notation and whether one could calculate the exact definite integral. Act 4 began with the notation of definite integrals, including the symbol of integration and a detailed description of each component. The text connected this notation back to the definition in Act 3 and the idea of a function being integrable if the Riemann sums of f converge over the interval $[a, b]$. This invited the question, "What are integrable functions?" (Q6) (ACT5). Additionally, Act 5 transitioned into discussion on how we know a limit exists and the existence of definite integrals. The text explained that this occurs if a function is continuous. In the previous lessons on limits and derivatives, there was a discussion on continuous functions and, therefore, the reader did not question what makes a continuous function. The reader now knew that continuous functions were integrable, but the text did not provide specific examples (ACT5). This flowed into Act 6's discussion on nonintegrable functions, or discontinuous functions, with an example of a piecewise function.

Acts 4 - 6 elaborated and built on the definition of the definite integral as the limit of Riemann sums. These acts added more details and filled in some missing information for the reader. The example in Act 6 did not ask explicit questions. As a result, the example informed the reader and more than challenged the reader to think and make connections with the discussion of continuous and discontinuous functions and integrable and non-integrable functions.

Act 7 progressed the story into "Properties of Definite Integrals" (p. 373) including the "order of integration, zero width interval, constant multiple, sum and difference, additivity, max-min inequality, and domination" (p. 374). In revealing all of these properties, the discussion only pertained to the order of integration and the zero-width interval. Missing from the discussion was perhaps the most important element, additivity, in assisting students in making connections to integration in contexts outside of mathematics (Jones, 2015; Jones & Fly, 2022; Oehrtman & Simmons, 2023) Continuing the discussion of integration, the text provided an explicit question: "What would happen if we instead move right to left, starting with $x_0 = b$ and ending at $x_n = a$?" (p. 373) (Q8). The reader had little time to process the question with a quick reveal of the answer. However, an implicit question formed while reading: "What happens if the integral has zero width?" (Q9). There was even less time to process this question unless one stopped reading to think about it, as the text revealed the answer to the integral 0, in the next line of the text. The text then presented the reader with examples using the rules in Acts 8 and 9. In Act 8, three different questions: "Suppose that $\int_{-1}^1 f(x)dx = 5$, $\int_1^4 f(x)dx = -2$, $\int_{-1}^4 f(x)dx = 7$. Then $\int_4^1 f(x)dx = \dots$, $\int_{-1}^1 [2f(x) + 3h(x)]dx = \dots$, and $\int_{-1}^4 f(x)dx = \dots$?" (p. 376) (Q10, Q11, Q12), required the use of the properties. Act 9 asked a higher-level question requiring the reader to do more than simple computations by following some properties: Can you "show that the value of $\int_0^1 \sqrt{1 + \cos x} dx$ is less than $\frac{3}{2}$?" (p. 376) (Q13).

In Act 7, the text listed many characteristics of the main character of the story, the definite integral. The reader had several properties to use when solving problems involving definite integrals. The text labeled each rule and used mathematical notation for each rule. Additionally, images in the first quadrant of a Cartesian plane represented the rules graphically or visually for the reader. However, the text only explained the first two rules. The reader found all the other rules in a table. The text later proved the max-min inequality property, while leaving the others for the reader to prove or find proofs of elsewhere if the reader wanted to know how to prove the rule. Acts 8 and 9 required the reader to understand and apply some of the properties revealed in Act 7.

Act 10 focused on the “Area Under the Graph of a Nonnegative Function” (p. 376). The subheading of Act 10 invited the reader to ask the following question: What is the area under the graph of a nonnegative function? (Q14) and noted that if the reader could answer this question, they understood the discussion of the definite integral. Act 11 challenged the reader to make sense of the definition of the definite integral as area under a curve of a nonnegative function and to utilize Riemann sums in searching for the answer to the questions: What is $\int_0^b x dx$? (Q15) and What is the “area A under $y=x$ over the interval $[0,b]$, $b>0$ ” (p. 377)? (Q16). The answer to question 15 (ACT12) provided the work to answer question 16 (ACT13). Act 12 was more theoretical and applied to the understanding of the definite integral as the limit of Riemann sums to arrive at a solution of $\int_0^b x dx = \frac{b^2}{2}$. Act 14 wrapped up the solutions to Questions 15 and 16 and stated two additional rules for computing definite integrals based on those solutions.

Acts 10 - 14 draw more connections and conclusions about the area under the graph of nonnegative functions. Questions explicitly asked and then answered led to additional properties of definite integrals and the addition of more characteristics of the main character of the story. The acts continued to add details and understanding of the definite integral.

The story ended with Acts 15 and 16 and the connection between integral notation and a prior discussion of the average value in the previous lesson. Act 15 extended prior knowledge of the average value of the area under the curve of a nonnegative function to a more general rule for any function. The discussion led to the question: How does $\text{Average} = \frac{1}{b-a} \int_a^b f(x) dx$ apply to any function? (Q17), but the text did not fully answer this question. A diagram in the text could have helped to explain some of the answer to the question, but the reader never found out the complete answer. The text challenged the reader to find the average value of the function $f(x) = \sqrt{4 - x^2}$ on $[-2,2]$. So, while it remained unclear perhaps how this applied to any function, the text expected the reader to apply it to the given function, a nonnegative semi-circle. The text computed the integral by finding the area under the curve geometrically (ACT16).

Acts 15 and 16 built on prior knowledge, revealing the notation with integrals for average value. However, the methods for finding the average value appeared to be the same as in the previous section except for the use of the integral notation. The text expected the reader to use geometric methods to find the integral of the non-negative semi-circle, which brings into question why the text revisited average value, instead of waiting for another section where the reader would finally know how to compute definite integrals using the FTC.

Rogawski & Adams (2015) "The Definite Integral" Mathematical Plot Diagram and Vignette

"The Definite Integral" lesson found in Chapter 5, "The Integral," in *Calculus*, 3rd edition, follows section 5.1 "Approximating and Computing Area" and proceeds the others sections, including "The Indefinite Integral," "The Fundamental Theorem of Calculus, Part 1 and II, "Net Change as the Integral of a Rate of Change," and "Substitution Method." Figure 6 depicts the mathematical plot diagram of "The Definite Integral" lesson. The diagram contains the story arcs numbered as they appear in the lesson and breaks down the questioning style and reveal of the answers of each story arc, or question. The collection of shaded cells forms the story of the lesson.

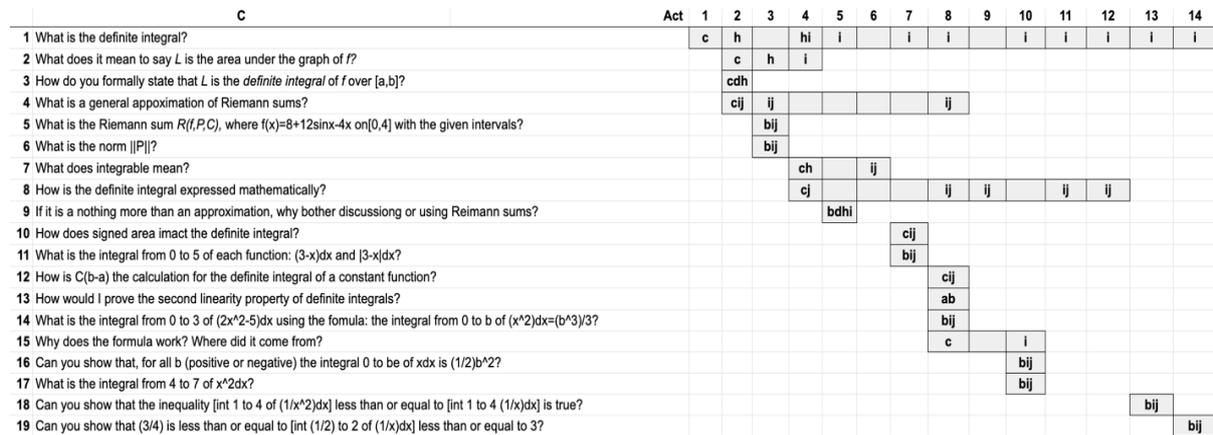


Figure 6. Mathematical Plot Diagram of Rogawski & Adams (2015)

Looking at the mathematical plot diagram of "The Definite Integral" lesson in Figure 6, the overarching story arc, What is the definite integral?, spans all 14 acts as the answer slowly revealed a little at a time within each act and as the other story arcs are developed and completed. There are two other story arcs, What is a general approximation of Riemann sums? and How is the definite integral expressed mathematically?, that span multiple acts and build from additional story arcs. Otherwise, the story arcs are short, indicating that the story quickly asks and answers questions leaving little time for the reader to think through the answer on their own. The vignette of this lesson provides a more detailed glimpse and breakdown of the story than the mathematical plot diagram.

The lesson began with the implicit question: What is the definite integral? (Q1) based on the title of the lesson (ACT1). Act 2 connected the previous lesson concluding that "if f is continuous on an interval $[a,b]$, then the endpoint and midpoint approximations approach a common limit L as $N \rightarrow \infty$ " (p. 237) with the notion of a L being the definite integral. The text left this statement unaddressed, inviting the implicit question: What does it mean to say L is the area under the graph of f ? (Q2) and How do you formally state that L is the definite integral of f over $[a,b]$? (Q3). The focus shifted to Riemann sums, or more general approximations, and left the reader in suspense. The shift triggered another implicit question: What is a general approximation of Riemann sums? (Q4). When explaining the more general form of the the approximations, the reader had the opportunity to try it with the function $f(x) = 8 + 12 \sin x - 4x$ on $[0,4]$ with a variety of given partitions (Q5) (ACT3). The text provided a discussion around the solution to Question 5 and the story shifted back to focusing more specifically on the definite integral with the idea of integrable, how to express the definite integral, and an explanation of the notation

(ACT4). Act 5 then went on to ask why introduce Riemann sums if we have the FTC for definite integrals. The text gave a brief explanation in regards to the importance of Riemann sums in its theoretical role and in proofs, not so much computational. The text promised to revisit and further study Riemann sums (ACT5). Act 6 answered the question: What does integrable mean? (Q7) by stating if f is continuous over a given interval, then it is continuous over the same interval.

Answering the implicit question, What is the definite integral?, became the overarching question of the lesson and the goal of the story. Acts 2 - 6 laid the foundation for discovering and understanding the definite integral through the connection to endpoint and midpoint approximations and Riemann sums. Upon making connections with the approximations, the text presented the definite integral notation and described to the reader the connection between the definite integral and the approximation of the area under the curve, a previously studied concept. Several concepts are connected here, but understanding integrals as approximations with Riemann sums and area under the curve were found by Jones (2015) to be less productive understandings when transferring mathematical understanding to contexts of physics.

In Act 7, the story transitioned from relating approximations to the definite integral to signed areas in relation to the definite integral, inviting the implicit question: How does signed area impact the definite integral? (Q10). The text quickly invited the reader to calculate two integrals using the idea of a signed area and its relation to the definite integral. The answer formulated through the use of geometric methods and the understanding of positive, or above the x -axis, and negative, or below the x -axis, areas.

Act 7 continued with previous connections to the definite integral with a discussion on signed area. The text explicitly invited the reader to demonstrate their understanding with an example involving two different functions, one of which was an absolute value function, which made the function nonnegative and, therefore, all area above the x -axis, or positive area.

In Act 8, the story began to develop characteristics of the definite integral with several properties and continued this into Act 11. The text quickly introduced the first two properties, “Integral of a Constant” and “Linearity of Definite Integrals” (p. 240) (ACT8). The linearity properties included proofs connecting back to Riemann sums. The text then asked the question: What is “ $\int_0^3 (2x^2 - 5)dx$ using the formula $\int_0^b x^2 dx = \frac{b^3}{3}$?” (p. 241) (Q14). While this allowed the reader to use the linearity property, the reader wondered: Why does the formula $\int_0^b x^2 dx = \frac{b^3}{3}$ work? (Q15). Act 9 focused on the properties of changing the bounds or having the same bounds, which provided more ways to express the definite integral and continued to build on answering Question 8. Act 10 provided the reader with an opportunity to show their understanding of definite integrals through answering the following question: Can you show “that, for all b (positive or negative), $\int_0^b x dx = \frac{1}{2}b^2$?” (p. 242) (Q16). This question connected back to the formula given previously in this lesson for the integral of x^2 . The solution followed up the question in the text, which then shifted to Act 11 involving the “Additivity for Adjacent Intervals” (p. 242) property. The reader then used this property along with the proven formula, $\int_0^b x dx = \frac{1}{2}b^2$, to answer the explicit

question: What is “ $\int_4^7 x^2 dx$?” (p. 242) (Q17).

Acts 8 through 11 followed a similar structure by introducing a property and then stating an explicit problem to solve using the properties. Throughout the introduction and use of the properties, the explanations were always related to the area under the curve. The text and the reader further developed the relationship and understanding of the definite integral as the area under the curve throughout these acts.

Act 12 continued introducing one more property, or theorem, the “Comparison Theorem.” The reader then quickly put the theorem into practice with examples in Acts 13 and 14. The final questions of the lesson involved using the comparison theorem: Can you show that “ $\int_1^4 \frac{1}{x^2} dx \leq \int_1^4 \frac{1}{x} dx$ ” (C, 2015, p. 243) and “ $\frac{3}{4} \leq \int_{\frac{1}{2}}^2 \frac{1}{x} dx \leq 3$?” (p. 243) (Q18, Q19). The text provided proofs of both of these inequality statements.

Acts 12 - 14 wrapped up the properties and challenged the reader to use the final property, the “Comparison Theorem.” Questions 18 and 19 were not as straightforward as the previous explicit questions about the properties and provided an opportunity for the reader to make deeper connections while still connecting the definite integral back to the area under the curve, but missing in this story is a connection to the notion of integration as accumulation, what researchers have found to be of most significance when transferring understanding in application (Nguyen & Rebello, 2011; Oehrtman & Simmons, 2023; Von Korff & Rebello, 2012).

Understanding the Curriculum from the Narrative Framework

When reading the titles of the lessons, one might think that the lessons were very similar. Although the findings did reveal an overarching theme of working towards defining the definite integral, there were also key differences between the lessons. Each lesson analyzed introduced the key question, What is the definite integral? early and developed its answer throughout the lesson. However, the path to answering this question varied from lesson to lesson as some focused on the area model and others utilized the FTC or introduced properties and formulas to compute definite integrals. Additionally, while all five lessons posed or implied a similar number of questions and, therefore, a similar number of story arcs, the sequencing and types of questions varied along with the connections made to help students develop and understanding of the definite integral. The questions varied in their formulation, as well as in the disclosure of the answer. The text answered some questions right away, and other questions left the reader wondering for multiple acts or even throughout the entire story.

Characteristics within Acts

Each lesson had a unique sequence of acts as revealed in the mathematical plot diagrams and the vignettes, and therefore a unique story. Goldstein et al. (2010) was possibly the most unique of these lessons. The reader started the lesson wondering about the definite integral, and then the lesson immediately shifted to understanding the FTC and its relation to finding the definite integral. The lesson addressed positive and negative functions with FTC, and the text asked the reader to work through problems utilizing FTC to find definite integrals. About three-quarters of the way through the lesson, the text questioned the connection with the FTC and area under the curve.

The lesson then ended with application problems using definite integrals and the area under the curve in which the understanding of an integral as the area under the curve made sense. The two application problems took two different approaches. First asking the reader to use the FTC and then asking about area under the curve and visa-versa. However, this is not representative of all application problems as students often struggle when transferring their mathematical knowledge to other contexts (Jones, 2015; Nguyen & Rebello, 2011).

Although Goldstein et al. (2010) introduced the reader to the FTC, there were no formulas to calculate the definite integral, which created a uniqueness in the Rogawski & Adams (2015) lesson, where the text introduced the formula $\int_0^b x^2 dx = \frac{b^3}{3}$ for finding the integral of a second-degree term. However, while the text stated the formula, the text did not explain why the formula worked. The missing explanation could create confusion for the reader as the reader may miss the connection. Thus, the implicit question surrounding why the formula worked remained unanswered. Further studies into future lessons would be needed to see if this question does eventually become answered as this is a necessary understanding for students to have if transferring their understanding of integration outside of a mathematical context. However, the Rogawski & Adams (2015) story did not start with the formula. Similarly, to all of the lessons, the initial question remained the same: *What is the definite integral?* From this question, the lesson referred back to Riemann sums and finding the limit as n approaches infinity of Riemann sums and compared this with the area under the graph. The text used the formula for Riemann sums to approximate the area under the curve. The explicit connection to Riemann sums and the definite integral remained mysterious to the reader, particularly since Riemann sums are only an approximation where the definite integral finds the exact area. This could add to the confusion of where the formula came from or why the formula works the way it does. Additionally, the reader needed to use the formula even though there was likely still uncertainty about the formula. The story ended with inequalities comparing definite integrals.

Similarly, to Rogawski & Adams (2015), Hughes-Hallet et al. (2009) started with a focus on Riemann sums while exploring why and how to construct sums for any function f . This discussion transitioned to the taking the limit of sums as n goes to infinity, also similar to Rogawski & Adams (2015). However, the Hughes-Hallet et al. (2009) lesson then spent some time exploring the accuracy of a desired integral while the reader worked to find the Riemann sums of integrals with various numbers of subintervals, including $n=10$ and $n=250$, to help build understanding. This set up the transition to finding the exact area, where the text referred to using a calculator or computer to calculate the value. This left the reader to potentially wonder how to calculate the exact integral, but should have developed an understanding of the definite integral as the area under the curve by the conclusion of the lesson. Nevertheless, the understanding of the definite integral as the area under the curve is not complete, but does help build their understanding of an integral as a measurable quantity, which is an important foundation in understanding the definite integral according to Oehrtman and Simmons (2023).

Additionally, the area under the curve was also significant in the Weir et al. (2008), as the lesson briefly started with finding the limiting value of a function in reference to Riemann sums. In this lesson, the focus shifted to determining if functions are integrable by defining integrable and nonintegrable. After the reveal and clarification of the definitions, the reader used the given definite integrals to find other definite integrals made up of a combination of the given segments of integrals. All of these definite integrals were of nonnegative functions.

Interestingly, the text situates the average value of a function, a uniqueness to this lesson, with integral notation. This provides the reader with an opportunity to apply their understanding of the average value to the function $f(x) = \sqrt{4 - x^2}$ on the interval $[-2, 2]$. However, the text expected the reader to use the area under the curve to find the definite integral, or in this case, the definite integral component in finding the average value of a function.

On the contrary, while Dietiker et al. (2017) laid the foundation to discover the definite integral from the title of the lesson, “How can I calculate the exact area?” (Dietiker et al., 2017), the approach was also unique. The lesson began with the Frieda and Fredo velocity and distance graphs, which the reader first saw in Lesson 1.5.1. This time the text challenged the reader to see the relationship between the two graphs and how the data for both existed in both graphs, just through different representations. The text then guided the reader through this particular comparison by directing the reader to explain the connection between slope and area. Dietiker et al. (2017) was the only lesson that referred back to slope and derivatives and more specifically made the connection between slope and area. The discussion on area quickly evolved into using rectangles and explaining how Riemann sums make an approximation of area.

However, the Riemann sums' introduction was brief and to the point without all of the formal mathematical notation that the other textbooks included. The text established the main idea of using Riemann sums to find the area of rectangles to approximate the area under the curve, along with the idea that the smaller the rectangles, the more accurate the area. This provided a transition as the lesson shifted to the formal definition of the definite integral: “the limit of a Riemann sum” (Dietiker et al., 2017). The text then introduced the definite integral notation, and the reader had the opportunity to demonstrate their understanding with specific explicit questions relating to the notation. The lesson ended with the reader finding the definite integral of four functions by examining the graph and using a limit of a Riemann sum or some other geometrical method. The answers to these questions were not immediately found in the text and provided the reader with the opportunity to challenge themselves and discover the answer. Dietiker et al. (2017) was the only text that left explicit questions unanswered for a period of time. Perhaps the slow reveal of answers to the questions allowed students to make connections across the ideas presented of slope, area, Riemann sums, and integral notation, which is important in working towards an understanding where integration in a math context could be transferred to other contexts. All other texts seemed to quickly answer explicit questions except for those involving overarching fundamental ideas, such as the following: What is the definite integral?

Characteristics of Story Arcs

The quick reveal or delayed reveal of answers is clear from the mathematical plot diagrams revealing the length of each story arc and the sub-coding in the story arcs. Specifically, the number of story arcs for all lessons was between 16 and 22. Several lessons had a handful of story arcs that span nearly all, if not all, acts of the lesson. These questions were directly related to the overarching theme of all the lessons about discovering and understanding the definite integral. Table 2 provides some specific data about the story arcs of the five stories of the definite integral.

Table 2. Characteristics of Story Arcs

	Hughes-Hallet et al.	Dietiker et al.	Goldstein et al.	Weir et al.	Rogawski & Adams
Percent of story arcs extended beyond one act	68%	45%	35%	44%	32%
Average length of the story arc (in number of acts)	4.7	2.5	3.4	4.3	3.3
Percent of acts in story arcs with one or more codes	29%	56%	46%	27%	37%

The data found in Table 2, along with the previous discussion of the lessons, including the mathematical plot diagrams and vignettes, provide evidence of the differences between the stories. Rogawski & Adams (2015) had the least percentage, 32%, of story arcs extending beyond one arc and less than half of the percentage, 68%, of Hughes-Hallet et al. (2009) story arcs. This contrast in the data reveals that Rogawski & Adams (2015) answered questions more quickly than the Hughes-Hallet et al. (2009) lesson. This also indicates that the reader of the Hughes-Hallet et al. (2009) story had more time to process and dwell on the questions throughout the story.

Also, from the plot diagrams in Tables 2 – 6 we saw that all stories had some very short story arcs, and Table 2 revealed that the story arcs averaged the shortest duration in the Dietiker et al. (2017), Rogawski & Adams (2015), and Goldstein et al. (2010) lessons. These shorter story arcs often involved explicit questions asking the reader to solve a definite integral or interpret the meaning of the area under the curve, Riemann sums, properties of integrals, or even the FTC. Although the Dietiker et al. (2017) lesson averaged short story arcs, it, along with the Hughes-Hallet et al. (2009) lesson, contained a cluster of story arcs spanning more than one act toward the middle of the lesson, which the other lessons did not. The story arcs that spanned more acts left the reader thinking for a longer duration, as the lesson did not reveal the answers quickly. The questions of almost all lessons that spanned multiple acts resulted from proposals in the text or implicit questions. The text slowly revealed the answers to these questions as it explained and discussed additional information.

Additionally, while Dietiker et al. (2017) had story arcs that averaged the shortest duration, that story had the highest percentage, 56%, of acts in story arcs with more than one code. The multiple codes within the acts in story arcs indicate that the story did not just ask a question or reveal an answer, but the story further developed the answer to the question through offering a partial answer, which was often the situation in the Dietiker et al. (2017) story. Other common codes that appeared in the other stories included a promise and an explicit or implicit question within the same act or a suspended answer in addition to a partial answer. The stories in the Weir et al. (2008) and Hughes-Hallet et al. (2009) lessons had the least percentage, 27% and 29%, respectively, of acts in story arcs with more than one code, although this number is slightly skewed by the overarching story arcs that spanned nearly all acts, if not all acts, while working to answer: *What is the definite integral?* or some variation of understanding the definite integral, the main idea of the lesson.

On the other hand, the story in Goldstein et al. (2010) had 46% of the acts in story arcs with more than one code.

However, the Goldstein et al. (2010) story was straightforward and to the point with explicit questions and quick disclosure of answers after the first few initial questions setting up the lesson. But the quick posing and answering of the question remained within one act, and, therefore, the higher percentage of acts in story arcs with more than one code. As a result, the story arcs were short and only occasionally spanned multiple acts, as evident in the 35% indicated in Table 2. There were a few longer story arcs, but those all revolved around understanding the definite integral, and more specifically, the FTC.

Discussion

In studying the research question, How do calculus textbooks tell the story of definite integrals? I found variations in how the fundamental concept of the definite integral was developed in the five textbooks studied. As seen in the vignettes and the mathematical plot diagrams of each lesson, within each of the definite integral lessons, there was a key overarching question, and many lessons had a few additional significant questions spanning much of the story depending on the topics explored to develop the definite integral understanding. All authors sought for the reader to develop an understanding of the definite integral. Some lessons provided or even challenged the reader to wonder and think more independently to make the connections between prior and newly revealed knowledge, while other lessons provided direct answers and strategies for solving the question asked of the reader. In addition, other lessons incorporated a combination of questioning strategies and even left the reader thinking about small details that were not revealed.

However, these thoughts about the finer details, like why the formula provided worked for finding the definite integral algebraically or how the authors of the text arrived at an answer, might be areas of tension for the reader and the definite integral. These areas of tension could be avoidable and unnecessary in building an understanding of the definite integral, or perhaps they were intentional. However, unnecessary confusion could frustrate the reader and prevent the reader from seeing the larger picture. But a bigger picture could be needed to understand the benefits and limitations of the authors' choices around the areas of tension. For instance, while it is worthwhile to be able to algebraically find a definite integral, is it necessary at this point to simply have a formula to use when later on the reader will learn the rule and, more importantly, the connection to finding a derivative? Maybe with the formula, $\int_0^b x^2 dx = \frac{b^3}{3}$, the reader would make the connection to derivatives. This is one possible place of tension in one of the stories that I have tried to make sense of, but it may not cause others tension, as the story is personal and subjective based on prior experiences and knowledge, no matter how much one tries to separate oneself from their experiences and knowledge. Another possible place of tension might be the exact answers to the definite integral that did not include explanations. Was it necessary to provide no explanation? Or was the main idea to continue building the connection between the definite integral and the area under the curve? Further research and analysis of the stories before and after this specific story, as well as the story of the entire unit, could possibly clarify these questions.

However, these moments of tension along with the entirety of the narrative analysis of the lessons revealed a great deal of the development of the main character, the definite integral. The goal of this research was not to show or prove one way better than the other, but to write the story as told by the text from the reader's point of view. In

doing so, a variety of stories appeared, each of which offers unique elements to the story but also which connect back to earlier stories, including the stories of derivatives and, more specifically, Riemann sums and limits, and could link to future stories that were not explored or visited.

Implications

Taking the time to read multiple lessons of a single textbook as a story could offer new information to teachers and students. Writing and analyzing the various stories of definite integrals challenged me to think about what story my lessons on definite integrals revealed. Although the textbook used in my classroom was not involved in my study, as it was an AP Calculus textbook, the various stories of the definite integral contributed to my lesson on definite integrals while working toward making connections between the area under the curve, Riemann sums, and the FTC. Upon reflection on my teaching of integration and the definite integral, I discovered that my textbook story evolved from ideas of antidifferentiation to geometric areas under the curve, Riemann sums, indefinite integrals, definite integrals through geometric areas under the curve, and finally to the FTC. The act of analyzing the stories of the definite integral changed my lessons, as I saw the various avenues in studying the definite integral and, therefore, impacted the learning of my students. Therefore, my findings on the stories of the definite integral and the analysis of the stories have implications both for teachers and students, as student learning is often a product of the teachers' lessons.

Implications for Teachers

Curriculum resources, particularly a textbook, if provided, often guide novice teachers' instruction. For me, during my first year of teaching calculus at the high school level, but in the fourth year of teaching, the textbook my school provided was my foundation for the course. Although I had a few colleagues that I could ask for some guidance, the textbook guided my instruction similarly as it had guided their instruction. Additionally, while my colleagues taught a few sections out of order, for my first year, I followed suit. It was not until in my third and fourth years teaching calculus that I began to see how my students were or were not making connections, and as a result I began to alter my course and even stray from the textbook. I wonder how my teaching would have changed if I had analyzed each of the lessons in the textbook as a story and begun to see the bigger story within each chapter.

Analyzing five other calculus textbook lessons on definite integrals allowed me to see and reinforce that there is not just one avenue to teach definite integrals and the development of prior knowledge and connections with the definite integral varies from text to text. These findings through this analysis and the previous findings of Grossman and Thompson (2008) about the significant impact of textbooks and curricular resources on teacher instruction reinforce the notion that teachers would benefit from opportunities to explore differences in curricular resources to determine what resources may best suit their teaching and the learning of their students. While analyzing a textbook through a narrative framework is only one of many possible methods to analyze a textbook, this analysis revealed similarities and differences across textbooks and drew attention to the inquiry process, the explicit or implicit questions, within each lesson or unit or even the textbook as a whole as the story

arcs represent the questions within the story.

When teachers or districts adopt textbooks, teachers should take the time to compare the different textbooks. However, completing this comparison can be the issue, as there is a need for teachers to better understand how to compare, or more so, analyze curricular resources, including textbooks. The narrative framework provides one way to compare textbooks, and similar to this study, one does not want one textbook to be better than the other, but rather to align the story of the lessons within the textbook with the learning needs of the students. For this reason, teachers need more research and examples to help them make sense of their findings and analysis method. Therefore, additional research and practice is significant, as this analysis can lead to a comprehensive understanding of the structure of the lesson and the textbook.

However, while using the narrative framework provides an analysis of the questioning and answering found throughout the lesson, it is time consuming. Instead of writing the whole story of each text lesson, teachers can find hints at the story in examining how the questions are presented and answered in the written curricular. Additionally, by looking at the reading lessons in the teacher's guide, teachers can examine the questions, both explicit and implicit, which result in the moment of inquiry in the lesson. Does the text provide an opportunity for students to continue to think through questions, or is the text a constant question-answer, question-answer? For instance, is the text causing the reader to draw their own conclusions or allowing the reader time to draw their own conclusions before revealing answers? This progression of asking and answering questions within the text indicates the potential questioning strategies that teachers utilizing the textbook will possibly incorporate into their instruction as they use the textbook as a guide or even a foundation for their course. Questioning strategies, among other parts of teaching, directly impact student learning. Therefore, in addition to the implications for students, implications for students are present as well.

Implications for Students

Student learning often results from the instruction of their teachers and their resources, including textbooks. If students read the textbook, the stories within the textbook will directly impact their learning. If the student does not read the textbook but relies on the instruction of the teachers, the stories within the textbook could indirectly impact their learning as the textbook often influences the instruction of the teachers. The variety of stories could appeal differently to students. As found in my analysis, some textbooks challenge the reader to ponder questions longer and form their own conclusions, whereas other textbooks ask the reader and then immediately answer the question. For instance, in Hughes-Hallet et al. (2009), several questions related to the notion of finding the sum for any function spanning any distance and sigma's relationship with definite integrals span the duration of the lesson. In Dietiker et al. (2017) the overarching questions spanning much of the lesson are on discovering what a definite integral is and how the exact area can be calculated. The development of these questions throughout the lessons is left unanswered for the students to think about. Yet, questions such as, *what is the integral from 1 to 4 of $\left[\left(\frac{1}{3}\right)x + \left(\frac{2}{3}\right)\right] dx?$* , in Goldstein et al. (2010) are answered immediately after being asked and thus eliminating wondering on the answer. These variances in the duration of the questions appeal to different learning styles and needs. Teachers who can identify these needs of their students and identify the variations in their written

curriculum can better meet their students' learning needs. Additionally, teachers must think of what comes next for their students. An understanding of integrals is imperative in other contexts and students need to be able to make connections to integration outside of calculus.

Conclusion

Through this study, I discovered the story of the definite integral from five calculus textbooks. Furthermore, this work contributed to further understanding how teachers can best select and use their resources while providing a different lens through which to read curriculum and providing an in-depth comparison on one individual lesson from five different calculus textbooks. The insights found in this study provide teachers, including myself, and researchers with a new perspective and a deeper understanding of these textbook lessons and the variances in a story that can alter student learning. However, exploring this one lesson may not allow for all the necessary components of understanding the definite integral and being able to transfer one's mathematical understanding of integration into a physical context as area and Riemann Sums preceded the official introduction of the definite integral and may indicate little contribution to students' sum-based understanding of definite integral, as needed according to Ely (2017). Additionally, it is important to consider Ely's (2017) question in regard to the calculus reform movement in increase application problems in textbooks, "are we explicitly equipping students with modes of interpreting calculus notation that support their modeling and interpreting in these application contexts...through specifically providing students with ways of interpreting the pieces of calculus notation so that they can reliably and meaningfully represent and work with quantities in the context at hand?" (p. 152).

Continued research is needed to discover more mathematical stories in more lessons and to contribute to the understanding of teachers of their curricular resources and, thus, impact student learning, as well as the development of new curricular resources. Additionally, additional research is needed to discover how the lessons implemented by teachers are similar to those in the written curriculum. This research and the findings could be extended even further if an analysis included the lessons enacted by novice and veteran teachers within a particular unit of a course. I know that my teaching of calculus in year one looked different from year five, and it would be interesting to see how aligned it remained with the structure of the lessons in the textbook as my textbook did not change. Finally, the additional research needed and the findings of this needed research could reinforce previous studies or reveal new information about the role that textbooks play in teacher instruction and student learning.

References

- Amos, N. R., & Heckler, A. F. (2015). Student understanding of differentials in introductory physics. In A. Churukian, D. Jones, & L. Ding (Eds.), *Proceedings of the 2015 physics education research conference* (pp. 35-38). American Association of Physics Teachers. <https://doi.org/10.1119/perc.2015.pr.04>
- Bal, M. (2009). *Narratology: Introduction to the theory of narrative*. (C. Van Boheemen, Trans.) (3rd ed.). University of Toronto.
- Barthes, R. (1974). *S/Z* (R. Miller, Trans.) Macmillan.
- Bressoud, D. M. (2011). Historical reflections on teaching the Fundamental Theorem of Integral Calculus. *The*

- American Mathematical Monthly*, 118(2), 99-115.
- Boyer, Carl B. (1959). *The history of calculus and its conceptual development: The concepts of the calculus*. Dover Publications, Inc.
- Dietiker, L. (2013a). *Framing a mathematics lesson as a story: A window into the aesthetics of a lesson*. [Paper presentation]. American Educational Research Association (AERA), California, United States.
- Dietiker, L. (2013b). Mathematics texts as narratives: Rethinking curriculum. *For the Learning of Mathematics*, 33(3), 14-19. <https://hdl.handle.net/2144/25242>
- Dietiker, L. (2015). Mathematical story: A metaphor for mathematics curriculum. *Educational Studies in Mathematics*, 90(3), 285-302. <https://doi.org/10.1007/s10649-015-9627-x>
- Dietiker, L., Brusoe, B., Hayes, J., Morrison, S. & Ray, M. (2017). *Calculus, Third Edition*. College Preparatory Mathematics Educational Program.
- Dietiker, L., Richman, A. S., Brakoniec, A., & Miller, E. R. (2016). Woo! Aesthetic variations of the “same” lesson. In Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.) *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tucson, AZ: The University of Arizona. <https://hdl.handle.net/2144/23705>
- Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior*, 48, 152-167. <http://dx.doi.org/10.1016/j.jmathb.2017.10.002>
- Ely, R. (2020). Teaching calculus with infinitesimals and differentials. *ZDM*, 53(3), 1591-604. <https://doi.org/10.1007/s11858-020-01194-2>
- Goldstein, L. J., Lay, D. C., Schneider, D. I., & Asmar, N. H. (2009). *Calculus & Its Applications, Twelfth Edition*. Prentice Hall.
- Grossman, P. & Thompson, C. (2008). Learning from curriculum materials: Scaffolds for new teachers?. *Teaching and Teacher Education*, 24, 2014-2026. <https://doi.org/10.1016/j.tate.2008.05.002>
- Healy, L. & Sinclair, N. (2007). If this is our mathematics, what are our stories? *International Journal of Computers for Mathematics Learning*, 12(1), 3-21. <https://doi.org/10.1007/s10758-006-9109-4>
- Hu, D., & Rebello, N. S. (2013). Using conceptual blending to describe how students use mathematical integrals in physics. *Physical Review Special Topics: Physics Education Research*, 9(2), article #020118. <https://doi.org/10.1103/PhysRevSTPER.9.020118>
- Hughes-Hallet, D., Gleason, A. M., McCallum, W. G., Osgood, B. G., Flath, D. E., Quinney, D., Lock, P. F., Rhea, K., Lomen, D. O., Tecoosky-Feldman, J., Lovelock, D., & Tucker, T. W. (2009). *Calculus: Single Variable, 5th Edition*. Wiley.
- Jones, S. R. (2015). Areas, anti-derivatives, and adding up pieces: Integrals in pure mathematics and applied contexts. *The Journal of Mathematical Behavior*, 38, 9-28.
- Jones, S. R., Lim, Y., & Chandler, K. R. (2016). Teaching integration: How certain instructional moves may undermine the potential conceptual value of the Riemann sum and the Riemann integral. *International Journal of Science and Mathematics Education*, 1-21. <http://dx.doi.org/10.1007/s10763-016-9731-0>
- Jones, S. R., & Ely, R. (2022). Approaches to integration based on quantitative reasoning: Adding up pieces and accumulation from rate. *International Journal of Research in Undergraduate Mathematics Education*, 9, 121-148. <https://doi.org/10.1007/s40753-022-00203-x>
- Larsen, S., Marrongelle, K., Bressoud, D., & Graham, K. (2017). Understanding the concepts of calculus:

- Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 526-550). National Council of Teachers of Mathematics.
- Leland, C. S., Burroughs, N., & Schmidt, W. H. (2015). Supporting classroom instruction: The Textbook Navigator/Journal. *Phi Delta Kappan*, *97*(1), 29-33. <https://doi.org/10.1177/0031721715602233>
- Netz, R. (2010). What did Greek mathematicians find beautiful? *Classical Philology*, *105*(4), 426-444. <https://doi.org/10.1086/657029>
- Nguyen, D., & Rebello, N. S. (2011). Students' difficulties with integration in electricity. *Physical Review Special Topics – Physics Education Research*, *7*(1), article #010113.
- Oehrtman, M., & Simmons, C. (2023). Emergent quantitative models for definite integrals. *International Journal of Research in Undergraduate Mathematics Education*, *9*(1), 39-64. <https://doi.org/10.1007/s40753-022-00209-5>
- Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, *14*(1), 1-18. <https://doi.org/10.1007/BF00704699>
- Pina, A., & Loverude, M. E. (2019). Presentation of integrals in introductory physics textbooks. In Y. Cao, S. Wolf, & M. B. Bennett (Eds.), 2019 PERC Proceedings (pp. 446-451). AAPT.
- Rasslan, S. & Tall, D. (2002, January). *Definitions and images for the definite integral concept*. [Paper presentation]. Psychology of Mathematics Education 26th Annual Meeting, Norwich, United Kingdom.
- Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*, 211-246. <https://doi.org/10.3102/00346543075002211>
- Remillard, J. T., Harris, B., & Agodini, R. (2014). The influence of curriculum material design on opportunities for student learning. *ZDM Mathematics Education*, *46*, 735-749.
- Richman, A. S., Dietiker, L., & Riling, M. (2019). The plot thickens: The aesthetic dimensions of a captivating mathematics lesson. *Journal of Mathematical Behavior*, *54*, 100671.
- Rogawski, J. & Adams, C. (2015). *Calculus, Third Edition*. W. H. Freeman & Company.
- Schoenfeld, A. H. (2001). Purposes and methods of research in mathematics education. In D. Holton (Ed.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study* (pp. 221-236). Kluwer.
- Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient? In S. Alatorre, J. L. Cortina, M. Sáiz, & A. Méndez (Eds.), *Proceedings of the 28th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 46). Mérida, México: Universidad Pedagógica Nacional.
- Sealy, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior*, *33*, 230-245. <https://doi.org/10.1016/j.jmathb.2013.12.002>
- Sherin, B. L. (2001). How students understand physics equations. *Cognition and Instruction*, *19*(4), 479-541. https://doi.org/10.1207/S1532690XCI1904_3
- Sinclair, N. (2004). The roles of the aesthetic in mathematical inquiry. *Mathematical Thinking and Learning*, *6*(3), 261-284. <http://www.sfu.ca/~nathsinc/MTLSinclair.pdf>
- Sinclair, N. (2005). Chorus, color and contrariness in school mathematics. *THEN: Journal* *1*(1). <https://www.edu.uwo.ca/mathstory/Sinclair.pdf>
- Steen, L. A. (1989). *Reshaping college mathematics: A project of the Committee on the Undergraduate Program in Mathematics*. Mathematical Association of America.

- Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26(2-3), 229-274. <https://doi.org/10.1007/BF01273664>
- van den Ham, A. & Heinze, A. (2018). Does the textbook matter? Longitudinal effects of textbook choice on primary school students' achievement in mathematics. *Studies in Educational Evaluation*, 59, 133-140. <https://doi.org/10.1016/j.stueduc.2018.07.005>
- Von Korff, J., & Rebello, N. S. (2012). Teaching integration with layers and representations: A case study. *Physical Review Special Topics: Physics Education Research*, 8(1), article #010125. <https://doi.org/10.1103/PhysRevSTPER.8.010125>
- Weir, M. D., Hass, J., & Giordano, F. R. (2008). *Thomas' calculus: Early transcendentals Update, 11th Edition*. Pearson.
- Westwood Taylor, M. (2016). From effective curricula toward effective curriculum use. *Journal for Research in Mathematics Education*, 47(5), 440-453. <https://doi.org/10.5951/jresematheduc.47.5.0440>

Author Information

Amanda Huffman Hayes



<https://orcid.org/0000-0003-0529-5373>

Purdue University

Beering Hall, #4127, 100 North University Street

West Lafayette, IN 47907

United States

Contact e-mail: huffma33@purdue.edu
