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To cite this article:

Valori, G., Albanese, V., Adamuz-Povedano, N., & Gómez-Torres, E. (2024). Effectiveness of paper folding and GeoGebra on students' geometric thinking and visualization. *International Journal of Education in Mathematics, Science, and Technology (IJEMST), 12*(5), 1395-1414. https://doi.org/10.46328/ijemst.4477

IJEMST

International Journal of Education in Mathematics, Science and Technology

2024, Vol. 12, No. 5, 1395-1414 https://doi.org/10.46328/ijemst.4477

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Article Info

Article History

Received:
05 April 2024
Accepted:
03 September 2024

Keywords

Geometric thinking
Visualization skills
Paper folding
GeoGebra
High school geometry

Abstract

Finding teaching methods that foster students' development of geometric thinking and visualization skills is still a current challenge. This study analyses the effects of the combined use of paper folding and dynamic geometry on high school students' geometric thinking and visualization skills. A quasi-experimental study pre-test/post-test design was conducted in distance learning in Italy. The instruments employed were van Hiele Geometry Test (VHGT) and the Visual Test (VT) before and after the treatment. The data were analysed using statistical tests according their nature. The results revealed that both groups have improved in the post-tests compared to their respective pre-tests. However, the students in the experimental group achieved better geometric thinking scores and performed better in three out of four sections of the visual test compared to the students in the control group. Thus, the proposed combination of instruments may be suggested in high school to make geometry teaching more effective, even with distance learning.

Introduction

Euclidean geometry is one of the most important areas of mathematical education, and its knowledge has been considered a virtue since antiquity. The acquisition of geometric competence provides students with the ability to formulate critical and logically coherent thoughts, argue, demonstrate, make choices and solve problems. In other words, it contributes to the development of what is nowadays referred to as students' mathematical literacy, which requires citizens to *reason mathematically*, rather than reproducing mathematical techniques as routines, to cope with an ever-changing world (OECD, 2018, p. 7). Despite its importance, geometry is regarded as a difficult subject and is not much appreciated by students (Jones, 2002; Mariotti, 2006). The dual nature of geometry - on the one hand a practical discipline linked to real-world problems and physical space, and on the other a theoretical field - constitutes its vital force but it is also the source of its teaching and learning difficulties.

Finding ways to teach geometry to support Thurston's "whole-mind approach to mathematical thinking" (cited in Jones, 2012), where geometric and spatial thinking are mutually nurturing, is a great challenge in mathematics education (2012). Many qualitative studies have shown that implementing tasks in dynamic geometry systems (DGS) can significantly improve students' geometry learning thanks to the dragging feature (Arzarello et al., 2002;

De Villiers, 1997; Laborde & Capponi, 1994; Mariotti, 2000), that allows “a fruitful interplay between the spatio-graphical and theoretical domains” (Laborde, 2005, p. 177). As a matter of fact, the benefits of using dynamic geometry systems in learning geometry have been confirmed by the results of many quantitative research studies (Jones, 2005; Ondes, 2021). The ancient art of paper folding Origami has also received increasing attention in secondary mathematics education. Introduced into primary school geometry teaching by Fröbel (Friedman, 2018), Origami began to be used for teaching geometry in secondary schools with Row's publication of “Geometric Exercises in Paper Folding" (Row, 1901). According to some studies, Origami instruction in secondary school significantly contributes to students’ success in geometry while increasing their reasoning and spatial thinking skills, self-efficacy and geometric habits of mind (Arıcı & Aslan-Tutak, 2015; Bornasal et al., 2021; Çakmak et al., 2014; Gür, 2020; Gürbüz et al., 2018; Kandil & Işıksal-Bostan, 2019). Some perspectives have recently opened up on the combined use of paper folding and dynamic geometry on different geometry topics (Budinski et al., 2018, 2019; Klemer & Rapoport, 2020; Sircar, 2015), but in-depth studies are still required on the efficacy of these practices on students’ learning of elementary geometry. Hence, this study aims to determine whether the aforementioned practices can enhance the development of geometric thinking and visuo-spatial skills.

Theoretical Background

Spatial Reasoning and Visualization in Geometry

The importance of spatiality in its various forms, which blend space, representational tools, and reasoning, is nowadays widely recognised in education (National Research Council, 2006; NCTM, 2000). Anyway, it is difficult to provide a single definition of spatial thinking since it has been a subject of interest for many disciplines, from neuroscience to cognitive psychology and mathematics education. A broad definition of spatial reasoning is as follows: “Spatial reasoning (or spatial ability, spatial intelligence, or spatiality) refers to the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects. Examples of spatial reasoning include: locating, orienting, decomposing/recomposing, balancing, diagramming, symmetry, navigating, comparing, scaling, and visualizing” (Mulligan, 2015, p. 513). Considering visualization as the capacity to “represent, transform, generate, communicate, document, and reflect on visual information” (Hershkowitz, 1990, p. 75), visualization and spatial reasoning overlap. The term visuospatial reasoning captures the connection between spatial reasoning and visualization and “emphasises the reasoning associated with and dependent on visual imagery but also expressed and argued with spatial references” (Owens, 2015, p. 7).

Psychologists have defined several spatial skills like spatial visualisation, spatial relations, spatial orientation, spatial perception, and mental rotation, and they have developed psychometric tests to measure them (Buckley et al., 2018). These tests can capture certain aspects of visuospatial reasoning; therefore, they are also used in mathematics education research (Arıcı & Aslan-Tutak, 2015; Boakes, 2006).

Solving a geometric problem involves moving between spatio-graphical and theoretical geometry (Laborde, 2005) since the role of diagrams and the complex nature of geometric figures, which have both conceptual and figural characteristics (Fischbein, 1993), must be taken into account. According to Duval (1995), there is a difference between visual perception and mathematical visualization. Although visual perception is one of the most important factors affecting the ability to recognize plane shapes, it only provides a direct access to the shape and never gives a complete apprehension of it. Duval (1994, 1995) distinguished four cognitive apprehensions: perceptual, sequential, discursive and operative. He stated that a diagram can acquire geometric significance if it evokes not only perceptual apprehension but also at least one of the other three types of apprehension (Duval, 1995, p. 143). Operative apprehension can provide an insight into a problem solution, by operating on the figure -physically or mentally- with specific figural processing: mereological, optical, or place modifications (Duval, 1995, p. 147).

Kalogirou et al. (2009) studied the interrelations between some spatial ability components and the understanding of geometrical figures, assuming a relationship between spatial ability and geometrical figure apprehensions in the sense given by Duval (1995, 1998). Kalogirou et al. (2009) argued that spatial ability is closely linked with perceptual and operative understanding of geometrical figures.

This study focuses on spatial visualization defined as “the ability to manipulate or transform the image of spatial patterns into other arrangements” (Ekstrom et al., 1976, p. 173) and on flexibility of closure, an important component of spatial ability (Carrol, 1993), defined as “the ability to hold a given visual percept or configuration in mind so as to dissembled it from other well defined perceptual material” (Ekstrom et al., 1976, p. 19).

The van Hiele’s Model of Geometric Thinking

The van Hiele model of geometric thinking, developed in the late 1950s (van Hiele, 1959/1984), is one of the best-known frameworks for studying the teaching and learning of geometry. The basic idea is that geometric thinking consists of first establishing connections among different real and abstract representations and then knowing, operating, reconstructing the “whole” system of relations among geometric concepts and the “theorems” that link them (van Hiele, 1959/1984).This model has produced considerable and ongoing research, including critical discussion of van Hiele's levels and its various components, such as sequentially, linearity and discretization of levels (Battista, 2007; Herbst et al., 2017; Jones & Tzekaki, 2016; Sinclair et al., 2017; Sinclair & Moss, 2012), as well as deep changes in the curricula of different countries around the world. For example, in the United States this theory has influenced the geometry strand of the NCTM Standards (NCTM, 2009, p. 55). According to the model, students’ geometric thinking develops through discrete and qualitatively different levels. The levels are sequential and hierarchical.

The level numbering used here is the same as that used by Hoffer (1981) and Usiskin (1982):

At level 1 (visualization), students compare drawings and identify, classify and order shapes according to imprecise qualities. They characterize the figures using visual prototypes, describe the shapes including irrelevant attributes and cannot conceive an infinite variety of shapes (Burger & Shaughnessy, 1986). Students sort shapes inconsistently and are not able to use properties as necessary conditions to determine a shape.

At level 2 (analysis), students compare shapes based on their properties even if they are unable to see class inclusions between certain types of shapes such as quadrilaterals. Additionally, they are not able to determine or characterize the shapes through a minimum number of sufficient conditions. At this level, students reject textbook definitions in favour of personal characterizations. They do not understand the meaning of mathematical proof and approach geometry empirically.

At level 3 (informal deduction), students understand the definitions of types of shapes and use them explicitly, sorting shapes according to class inclusions or other mathematically precise attributes. They use "if, then" statements correctly and construct informal deductive arguments, but do not distinguish the roles of axioms and theorems.

At level 4 (deduction), students use a precise language, frequently formulate conjectures, attempts to verify conjectures deductively, and recognise proof as the final authority for deciding the "truth" of a mathematical proposition. Within mathematical discourse, they recognize the roles of axioms, definitions, theorems, and proofs, implicitly accepting the axioms of Euclidean geometry (Burger & Shaughnessy, 1986).

At level 5 (rigor), students can work in different axiomatic systems, establish theorems and analyse/compare these systems (Fuys et al., 1988). At this level, students study geometries without resorting to concrete models (Burger & Shaughnessy, 1986).

The van Hiele theory outlines a progression of thinking intrinsic to scientific and mathematical thought, advancing through the phases of perceptualization, conceptualization, conceptual organization, and axiomatization. Geometry learning involves three main types of structuring: spatial, geometric, and logical. Spatial structuring corresponds to van Hiele level 1 and the transition to van Hiele level 2. Geometric structuring corresponds to van Hiele level 2, and logical structuring corresponds to van Hiele levels 3 and 4. Axiomatic structuring pertains to van Hiele level 5 (Battista, 2007).

Aim of the Study and Research Questions

The aim of this study is to examine the effectiveness of paper folding & GeoGebra laboratory activities in enhancing geometric thinking and visuo-spatial skills as opposed to traditional geometry teaching. The study was conducted in distance learning due to the COVID19 pandemic. The research questions are as follows:

1) Are there any differences between students instructed with Paper Folding & DGS-based activities and students instructed with traditional-based activities regarding the improvement of geometric thinking?
2) Are there any differences between students instructed with Paper Folding & DGS-based activities and students instructed with traditional-based activities in reference to visuo-spatial ability?

Method

Research Design

This study was a quasi-experimental research conducted during the emergency for the COVID19 pandemic in

Italy. The study used a pre-test-post-test design which involved the use of two groups of students, namely the control group and the experimental group. The control group was taught using conventional learning with alternating face-to-face and distance lessons; the experimental group was taught alternating traditional face-to-face and innovative distance learning lessons. The innovative learning lessons were structured as geometry laboratories which included activities with paper folding and GeoGebra.

In both groups, two questionnaires were applied before the instruction (pre-test) and the same questionnaires were applied after the instruction (post-test). The data collected were processed and analysed to determine whether there was an influence of the integration of regular instruction with Paper Folding and GeoGebra activities at distance learning on students' geometric thinking and visuo-spatial skills. The data analysis was carried out after obtaining data and was assisted using SPSS version 26.

Participants

In this quasi-experimental study, 96 Italian students (15-16 years old) of the general upper humanities secondary school in the Marche region, from two Institutes, were involved. A total of 52 students were in the treatment group and 44 students were in the control group.

Procedures and Treatment

During the treatment, the experimental group tackled plane geometry problems using paper folding and GeoGebra at distance learning, whereas the control group was instructed without paper folding and/or GeoGebra at distance learning. Both groups used Google suite for Education Apps at distance learning, and the experimental group used GeoGebra Classroom and GeoGebra Groups too. Both groups were taught by their teachers on the geometric topics foreseen in the curriculum and had the same opportunity to be engaged in classroom discussions. The design of the tasks involving paper folding and GeoGebra was made by the first author of this paper, who supported the teacher in the experimental classes during treatment (in monitoring activities and during class discussions). The paper folding problems were inspired by the specific literature (Haga, 2008; Hull, 2013; Johnson, 1957). Each task was submitted through an online worksheet (GeoGebra activity) with some guiding questions, posted in a GeoGebra group (first period of the treatment) or embedded in a GeoGebra class (second period of treatment). The decision to switch to GeoGebra classes was determined by their improved functionality compared with GeoGebra groups (which are currently no longer supported by the platform). The topics covered were triangles, quadrilaterals, Pythagorean theorem, and geometric constructions. The worksheet contained instructions to fold a two-dimensional origami model, usually accompanied by a diagram and a video and/or a digital simulation, which the students had to repeat using a sheet of paper (usually a square sheet cut from a common A4 printer sheet or a template) or instructions to explore by folding a geometric configuration. The same worksheet asked the students to supplement the physical activity with a digital activity. In addition, students were invited to build a digital model of the geometric configuration represented by folding using GeoGebra tools (or to explore a pre-build digital model, depending on the difficulty of the construction and their increasing mastery of the software).

For each folded model, there was one or more questions that students had to answer; some prompts invited students to make observations, formulate conjectures, and justify them using both physical and digital representations. The task design with such artifacts was conceived to favour the synergy of the cognitive processes involved in a geometric activity (Valori et al., 2024), i.e. the processes of construction, visualisation and reasoning (Duval, 1998) and mainly involved the figural, linguistic and numerical semiotic registers (Duval, 2017). The instructional treatment was carried out in two periods: the first in May 2020 and the second from November 2020 to March 2021. The total number of lessons in the treatment group was 14, excluding some lessons of introduction to the dynamic geometry software used, GeoGebra. The lessons in the experimental class lasted between 45 and 90 minutes, depending on the difficulty of the tasks, for a total of 15 hours and 45 minutes. Figure 1 shows an example of activity in the treatment, with geometrical processes and semiotic registers involved.

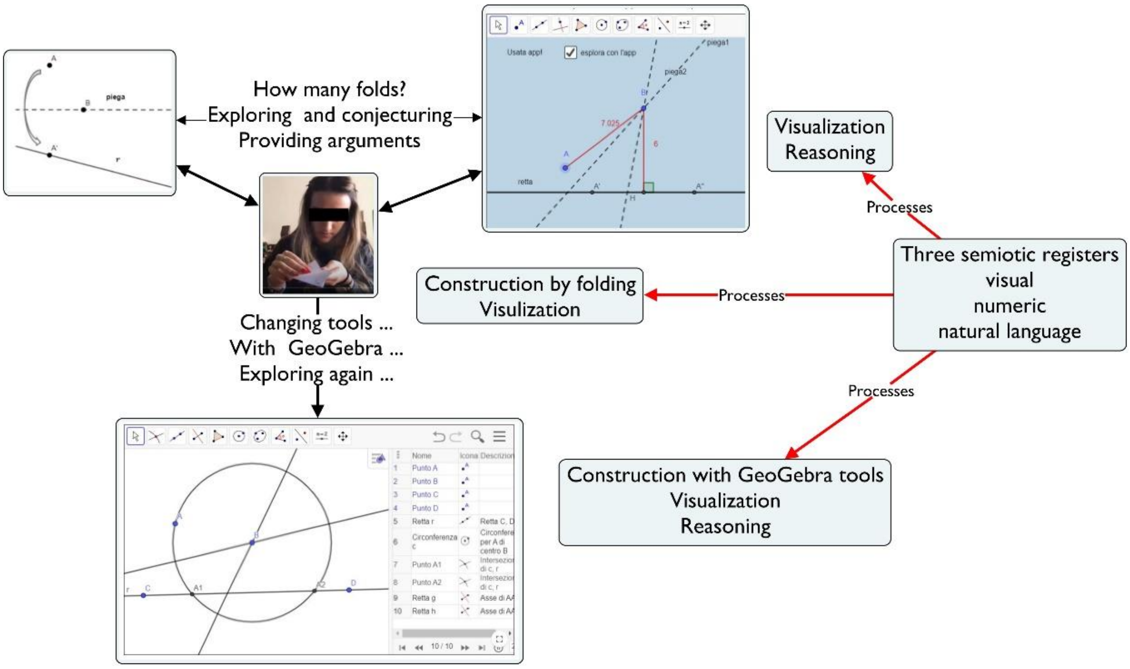

Figure 1. An Example of Activity

Instruments

The study used two tests to evaluate students' geometric thinking and visualization skills, respectively the van Hiele Geometry Test (VHGT) and the Visual Test (VT). The content validity of the VHGT was assessed by the authors and the teachers by comparing the questions with the National guidelines. The VHGT was developed by The Cognitive Development and Achievement in Secondary School Geometry project (CDASSG) (Usiskin, 1982). The VHGT comprised 25 items and for each level there was a cluster of 5 items. The test lasted 35 minutes.

This test was used accepting the basic ideas of van Hiele's model, but avoiding labelling students as belonging to a certain level. Hence, the test assessment was performed according to Usiskin's (1982) directions, but without assigning a level to each student. A level cluster was full-filled if the student answered at least three out of five items correctly, using Usiskin's (1982) weak criterion. The marking for each cluster was as follows: one point if the student passed Level 1 cluster, two points if the student passed Level 2 cluster, four points for Level 3 cluster, eight points for Level 4, and sixteen points for Level 5. The student's total score was the sum of the points obtained at each level cluster.

The Visual test (VT) presented 54 items and was built by assembling different tests from Ekstrom and his co-workers (1976) kit of factor referenced cognitive tests. The VT was built considering two factors: spatial visualisation and flexibility of closure. As regards the first factor, 12 items were selected from the Form Board Test- part 1 (VT1), 20 items from the Paper Folding Test- part 1 and part 2 (VT2) and 6 items from the Surface Development Test – part 1 (VT3); as regards the second factor, 16 items were selected from the Hidden Figures Test – part 1 referring to the flexibility of closure (CF1). The test lasted 28 minutes, according to the Kit instructions. The marking criteria of each section were adapted from the Kit (Ekstrom et al., 1976).

The total score of the VT1 section was the sum of the scores in each item, with a minimum of 0 and a maximum of 60. The total score of the VT2 section was the sum of the scores for each item, with a minimum of 0 and a maximum of 20. The total score of the VT3 section was the sum of the scores for each item, with a minimum of 0 and a maximum of 30. The total score of the CF1 section was the sum of the scores for each item, with a minimum of 0 and a maximum of 16.

Data Analysis

The data were analysed quantitatively. The scores in the VHGT and in the VT1, VT2, VT3, and CF1 sections of the VT were analysed with descriptive and inferential statistics using SPSS version 26. Descriptive analyses, goodness of fit tests, pair-sample tests, and independent sample tests were deployed to examine possible differences in students' geometric thinking and visualization skills, with a significance level (α) of 0.05. Different tests were performed according to the nature of each variable of interest.

Results

Thirty-eight students in the experimental group and 28 students in the control group responded to both the VHGT pre-test and post-test. Forty-one students in the experimental group (S) and 29 students in the control group (C) answered both the VT pre-test and post-test.

Geometric Thinking Results

To answer the first research question "Are there any differences between students instructed with Paper Folding & DGS-based activities and students instructed with traditional-based activities regarding the improvement of geometric thinking?", Table 1 presents the values of the mean, standard deviation, median, and range of the scores of the experimental and control groups obtained from the VHGT before the treatment. These scores were compared with Kolmogorov-Smirnov test to equal distribution. The K-S statistic was 0.224 (p-value 0.12); according to it, the groups' performance was similar because the differences between both VHGT scores' distributions were not statistically significant.

Table 1. Descriptive Statistics of VHGT Scores (Pre-Test; Post-Test)

	VHGT score pre-test		VHGT score post-test	
	Experimental	Control	Experimental	Control
n	38	28	38	28
Mean	3.79	2.68	4.76	3.36
S.D.	4.955	4.083	5.027	4.847
Median	2	1	3	1
Range	0-19	0-19	0-23	0-19

After the treatment, the students answered the VHGT test again. The descriptive statistics for the data obtained from the VHGT test after the treatment are also presented in Table 1; for both groups (experimental and control), an increase of the centrality measures of the van Hiele scores after the treatment was suggested. In order to recognize if these differences were statistically significant, three tests were applied. The post-test scores on the van Hiele test of the experimental and the control group were compared with Kolmogorov-Smirnov test to equal distribution. The K-S statistic was 0.352 (p-value 0.007), consequently there was a statistically significant difference in VHGT post-test scores between the groups. As these scores' distributions were asymmetric, to identify if variation between groups was linked to centrality measures, the Mann-Whitney test was used to compare median differences. The results [$U=351.5$; $p=0.007$] showed a statistically significant difference, which means that, after the treatment, the median of VHGT scores of the experimental group is statistically greater than the median of VHGT scores of control group. For each group, the Wilcoxon signed rank test was applied to compare the median of difference between scores before and after instruction. There was a statistically significant difference in the experimental group, with VHGT scores of the students after the treatment trending to be greater than before the treatment ($W = 210$; $p=0.013$). In contrast, there was no statistically significant difference in the control group, and the VHGT scores were statistically equal before and after treatment ($W = 138$; $p=0.222$). These results showed that the experimental group improved its global performance, but the control group did not.

A finer analysis was made on each cluster of the van Hiele test, and a marginal homogeneity test was applied to each group to compare performance by groups before and after treatment. This test analysed each cluster to determine if the distribution of the number of correct answers was equal before and after the treatment. In the control group (see Table 2), the distributions are statistically equal, which means that the performance of students was similar before and after the treatment in every cluster (visualization, analysis, informal deduction, deduction, and rigor in geometric thinking).

Table 2. Marginal Homogeneity Test for VHGT's Clusters (Control Group)

	L1	L2	L3	L4	L5
Distinct values	6	6	5	4	4
Cases outside the diagonal	21	21	19	21	20
MH statistic	68	42	20	17	16
p-value (bilateral)	0.338	0.629	0.329	0.276	0.504

In the experimental group (see Table 3), only there were statistically significant differences between the distribution of points of clusters L2 (p-value 0.005) and L3 (p-value 0.023), showing that the students of the experimental group trend to improve their performance, after the treatment, about levels of analysis and informal deduction in geometric thinking; however, the performance of students was similar before and after the treatment in levels of visualization, deduction, and rigor in geometric thinking (p-values greater than 0.05).

Table 3. Marginal Homogeneity Test for VHGT's Clusters (Experimental Group)

	L1	L2	L3	L4	L5
Distinct values	5	6	6	5	4
Cases outside the diagonal	26	28	27	23	21
MH statistic	93	68	29	20	19
p-value (bilateral)	0.889	0.005	0.023	0.337	0.163

To deepen these changes in the performance of the levels of analysis and informal deduction in geometric thinking, Table 4 and Table 5 show the mean, standard deviation, median, and range of the number of correct answers observed in the control group for L2 and L3, and the same Table 4 and Table 5 show the same statistics observed in the experimental group. To identify if the differences were linked to centrality measures, the Wilcoxon signed rank test was applied for each group.

Table 4. Descriptive Statistics for Analysis' Level in Geometric Thinking

	Experimental (n=38)		Control (n=28)	
Group	L2_PRE	L2_POST	L2_PRE	L2_POST
Median	2.5	3.0	2.0	2.0
Range	0-4	0-5	0-5	0-5
Mean	2.42	3.11	2.14	2.32
SD	1.130	1.331	1.533	1.188

Table 5. Descriptive Statistics for Informal Deduction's Level in Geometric Thinking

Group	Experimental (n=38)		Control (n=28)	
	L3_PRE	L3_POST	L3_PRE	L3_POST
Median	1.0	2.0	1.0	1.0
Range	0-4	0-5	0-2	0-4
Mean	1.24	1.71	0.93	1.18
SD	1.051	1.088	0.858	0.983

In the experimental group, there was a statistically significant difference between the number of correct answers for the analysis' level in geometric thinking, after the treatment, the number of correct answers increased (W = 326, p-value 0.001), as well as for informal deduction's level in geometric thinking (W = 288.5, p-value 0.006). In the control group, there was not a statistically significant difference between the number of correct answers for the analysis' level in geometric thinking before and after treatment (W=125, p-value 0.37), as well as for informal deduction's level in geometric thinking (W=122, p-value 0.153), as was expected for the marginal homogeneity test result (Table 2). Table 6 shows that 22 students in the experimental group improved their performance on questions about analysis' level in geometric thinking, 11 increased one point, 10 increased two points and one increased four points. Table 7 shows that 20 students in the experimental group improved their performance on questions about informal deduction's level in geometric thinking, 14 increased one point, 5 increased two points, and one increased three points.

Table 6. Number of Correct Answers for Analysis' Level (Experimental Group)

Number of correct answers pre-test	Number of correct answers post-test						Total
	0	1	2	3	4	5	
0	0	1	1	0	0	0	2
1	0	3	0	2	0	1	6
2	0	1	2	5	3	0	11
3	1	0	0	3	4	4	12
4	0	0	1	3	2	1	7
Total	1	5	4	13	9	6	38

Table 7. Number of Points for Informal Deduction's Level (Experimental Group)

Number of correct answers in pre-test	Number of correct answers post-test						Total
	0	1	2	3	4	5	
0	1	4	3	1	0	0	9
1	2	5	8	2	0	0	17
2	0	4	3	1	0	0	8
3	1	0	0	1	0	0	2
4	0	0	0	0	1	1	2
Total	4	13	14	5	1	1	38

Visuo-Spatial Ability Results

To answer the second research question "Are there any differences between students instructed with Paper Folding & DGS-based activities and students instructed with traditional-based activities in reference to visuo-spatial ability?", Table 8 presents the values of the mean, standard deviation, median, and range of the scores observed before the treatment from each section of the VT, as well as the results of the Lilliefors' test. The last showed goodness-of-fit to normality for most variables at a significance level of 5%, and only VT3 in the control group reject the null hypothesis of normality (p-value 0.037); in the experimental group, non-significance deviations of normality for VT1 (p-value 0.085) and VT3 (p-value 0.192) are explained by a slight right asymmetry of their scores' distribution. As consequence, nonparametric tests were used to compare the results of VT3 and normality-based tests for VT1, VT2, and CF1.

The pre-test VT1, VT2, and CF1 scores were compared with test of Levene to equal variances and the t-test to equal means. Table 9 shows the results. The null hypothesis of equality was not rejected in any case at 5% significance level. The pre-test VT3 scores were compared with Kolmogorov-Smirnov test for equal distribution; the K-S statistic was 0.222 (p-value 0.19). Accordingly, the groups' performance was similar because the differences between scores' distributions of the control group and experimental group were not statistically significant.

Table 8. Statistical Summary of VT Scores from Pre-Test

	Control group				Experimental group			
	VT1	VT2	VT3	CF1	VT1	VT2	VT3	CF1
n	29	29	29	29	41	41	41	41
Mean	14.10	5.76	3.41	2.69	13.73	7.07	4.29	3.44
S.D.	8.821	3.113	5.200	2.055	11.064	4.227	4.337	1.790
Median	14	6	1	3	9	7	3	3
Range	1-36	0-12	0-22	0-8	2-42	0-16	0-17	0-8
Lilliefors Statistic	0.114	0.093	0.256	0.174	0.192	0.152	0.165	0.133
p-value	0.805	0.946	0.037*	0.308	0.085	0.275	0.192	0.425

* Result statistically significant at 5%

Table 9. Comparison among VT1, VT2, and CF1 Scores Distribution (Pre-Test)

		Levene test		t-test	
	Mean difference	F	p-value (bilateral)	t	p-value (bilateral)
VT1	0.372	3.230	0.077	0.150	0.881
VT2	-1.315	3.202	0.078	-1.423	0.159
CF1	-0.749	1.104	0.297	-1,623	0.109

Table 10. Statistical Summary of VT Scores from Post-Test

	Control group				Experimental group			
	VT1	VT2	VT3	CF1	VT1	VT2	VT3	CF1
n	29	29	29	29	41	41	41	41
Mean	16.24	8.52	3.83	3.17	22.46	8.66	5.98	4.39
S.D.	10.957	4.085	4.520	2.465	13.860	4.470	5.863	2.155
Median	15	9	2	3	22	8	6	4
Range	1-46	3-16	0-14	0-13	1-52	2-17	0-23	1-9
Lilliefors Statistic	0.111	0.150	0.228	0.231	0.084	0.114	0.163	0.145
p-value	0.830	0.484	0.084	0.077	0.911	0.617	0.202	0.325

After the treatment, the students took the VT again; the descriptive statistics for the data are presented in Table 10. The comparison of Table 8 with Table 10 suggested an increase in the centrality measures after the treatment for both groups (experimental and control). Different tests were applied to determine if these increases were statistically significant. Table 10 presents the results of the Lilliefors' test for the experimental and control group after treatment from each section of the VT. All variables showed goodness-of-fit to normality at a significance level of 5%. The right asymmetry of the scores' distribution explained non-significant deviations of normality for VT3 in the control group (p-value 0.084) and in the experimental group (p-value 0.202). One right outlier observed in the scores of CF1 in the control group (p-value 0.077) contributed to the distance between CF1 scores distribution and normal distribution, without rejecting the null hypothesis of normality.

Again, nonparametric tests were used to compare the results of VT3 and normality-based tests for VT1, VT2, and CF1, because the VT3 scores distribution in the pre-test of the control group did not fit the normality distribution. The post-test VT3 scores of the experimental and control group were compared with Kolmogorov-Smirnov test to equal distribution. The K-S statistic was 0.241 (p-value 0.144); consequently, the difference in post-test VT3 scores between the groups was statistically nonsignificant. The post-test VT1, VT2, and CF1 scores of the experimental and control group were compared with the Levene test to equal variances and the t-test to equal means.

Table 11 shows the results. The null hypothesis of equality was rejected at 5% significance level only in two cases; thus, VT1 and CF1 scores of the experimental group trend to be greater than VT1 and CF1 scores of the control group. Accordingly, the groups' performance was similar for the VT2 and VT3 sections because the differences between the scores' distributions of the control group and experimental group were not statistically significant; however, the groups' performance was different for the VT1 and CF1 sections because the mean of the scores' distributions of the experimental group was statistically greater than the mean of the control group.

Table 11. Comparison among VT1, VT2, and CF1 Scores Distribution (Post-Test)

		Levene test		t-test	
	Mean difference	F	p-value (bilateral)	t	p-value (unilateral)
VT1	-6.222	2.478	0.120	-2.012	0.024*
VT2	-0.141	0.204	0.653	-0.135	0.446
CF1	-1.218	0.250	0.619	-2.194	0.016*

* Result statistically significant at 5%

For each group, the test for paired samples was applied to compare differences between scores before and after instruction. Table 12 shows the results of the t-tests for paired samples to compare VT1, VT2 and CF1 scores distribution of the experimental and control groups before and after instruction. In the experimental group, there were statistically significant differences for the three sections of VT, scores of the students after the treatment trending to be greater than before the treatment (last column of Table 12). In the control group, there were no statistically significant differences in sections VT1 (p-value=0.208) and CF1 (p-value=0.221), thus their scores were statistically equal before and after treatment; however, there was a statistically significant difference for VT2, scores of the students after the treatment trending to be greater than before the treatment (p-value=0.0005). These results showed that the experimental group improved its global performance, but the control group did so only in one section.

Table 12. t-Tests for Paired Samples (Before and After Instruction)

	Mean difference	Control group (n=29)		Experimental group (n=41)	
		t	p-value (unilateral)	t	p-value (unilateral)
VT1	-8.732	-0.825	0.208	-5.711	0.000*
VT2	-1.585	-3.708	0.0005*	-2.789	0.004*
CF1	-0.951	-0.780	0.221	-2.224	0.016*

* Result statistically significant at 5%

To deepen these changes, a graphic illustration was used to see the improvement in each section of VT that showed statistically significant differences. Scatterplots of the scores before and after the treatment showed the scores observed for each student; the dots above the identity function (diagonal line) correspond to students with better scores in the post-test than in the pre-test, the dots below the diagonal line correspond to students with better scores in the pre-test than in the post-test, the dots on the diagonal line correspond to students with equal scores in both applications.

Figure 2 shows scatterplots of the VT2 scores in the control and experimental groups. Both groups improved their performance after the treatment with respect to pre-test, the increases were similar in both groups.

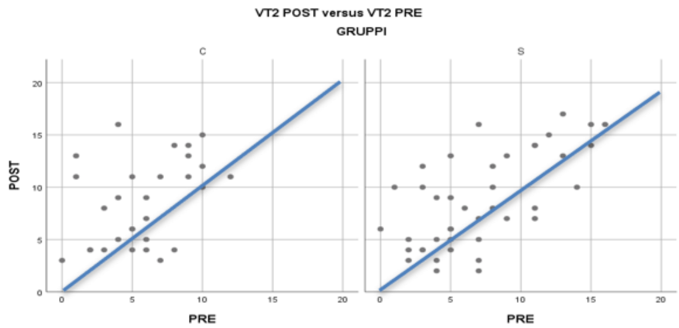

Figure 2. Scatterplots of VT2 Scores (Control-Left and Experimental-Right)

Figure 3 shows scatterplots of the VT1 and CF1 scores to the experimental group. This group improved his performance after the treatment with respect to pre-test.

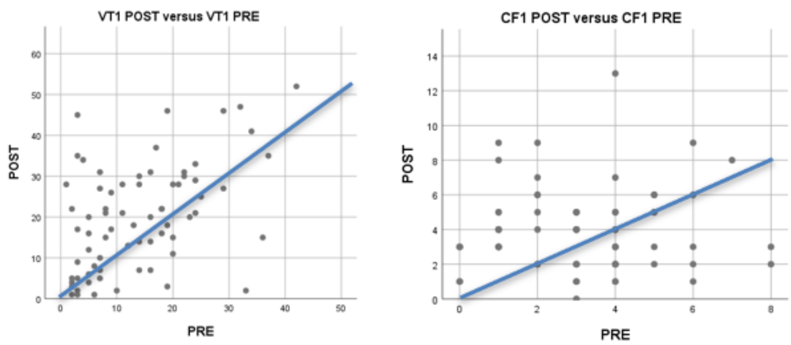

Figure 3. Scatterplots of VT1 and CF1 Scores (experimental group)

Finally, for each group, a marginal homogeneity test was applied to compare scores in the VT3 section before and after treatment. Accordingly, the distributions were statistically equal at a significance level of 5% in the control group (MH statistic was 98; p-value=0. 776) and in the experimental group (MH statistic was 170; p-value=0.098).

As VT3 scores' distributions were asymmetric, to identify if there was variation in each group, the Wilcoxon signed rank test was used to compare the median of difference between scores of pre-test and post-test. The results showed a statistically significant difference in the experimental group (W=394; p-value=0.050); but, in the control group did not (W=161.5; p-value=0.243).

The scatterplots are used to better understand the results of the VT3 section, considering that some patterns were difficult to identify for the tests because of the asymmetry of the VT3 score distributions. Figure 4 shows scatterplots of the VT3 scores for the control and experimental groups. The experimental group improved his performance after the treatment with respect to pre-test, both groups trend to low scores (lesser than 10).

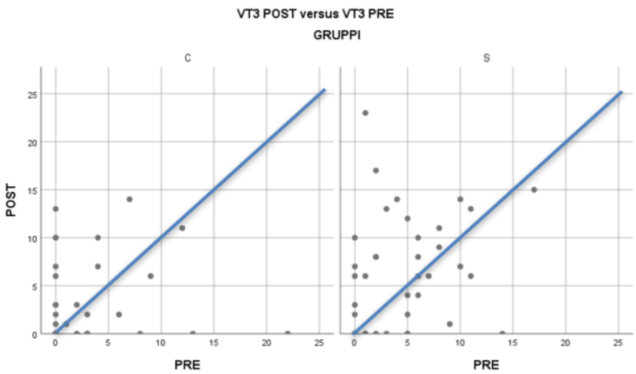

Figure 4. Scatterplots of VT3 Scores (Control-Left and Experimental-Right)

Discussion and Conclusions

As stated in the introduction, the aim of this study was to investigate the effect of using paper folding and DGS (GeoGebra) on high school students' geometric thinking and visualization skills. For this purpose, we conducted quasi-experimental research with pre-test post-test design involving two groups of students, an experimental group and a control group. Our results showed that the experimental group improved its global performance in the Van Hiele test after treatment, whereas the control group did not. The results of this study are consistent with studies that analyzed the use of paper folding (Arıcı & Aslan-Tutak, 2015; Bornasal et al., 2021) and the use of dynamic geometry (Idris, 2007; Kutluca, 2013) separately.

The results of a finer analysis on each cluster of geometric thinking (visualisation, analysis, informal deduction, rigor) showed that, after the treatment, the students of the experimental group improved only the scores of the analysis and informal deduction levels, while the control group did not. The improvement in geometric analysis and informal deduction may be the result of construction and observation activities performed by folding, as already pointed out by Golan (2011), and of modelling activities carried out with digital tools, explorations and validations by dragging, as highlighted by Gawlick (2005). Moreover, the combination of physical and digital artefacts, along with an environment that promoted group work and argumentation, helped the students to acquire a certain awareness of geometry, as already pointed out in Valori et al. (2024). The students understood that in geometric constructions, certain properties must be mobilized, that they can change by changing the tools and that every geometric statement in the form *if ..., then ...* expresses a logical dependence that can be read in terms of invariants. The exploratory activities with the formulation of conjectures favoured the production of crystallised statements of the form *if ..., then ...* and arguments too (typical van Hiele level 3 behaviours). Students were not ready to enter the higher van Hiele level 4. Nonetheless, during the treatment, some students performed in accordance with van Hiele level 4.

The improvement of students' geometric thinking after only 14 lessons is consistent with Dina van Hiele-Geldof's study, which required 20 lessons for the students to reach the second level of geometric thinking and approximately 50 for the next level (van Hiele-Geldof, 1957/1984). Although we did not use the same teaching methodology (van Hiele phases) as the van Hieles', some of the stages outlined in their theory were proposed in the laboratory activities. In particular, physical and digital materials were used as in the inquiry phase; the guided phase was evident in prompting students to develop arguments and explanations; the explication phase corresponded to class discussions, and the orientation phase involved tackling new and challenging problems.

In the visual test (VT), both groups improved the average scores in each section of the test (VT1, VT2, VT3, CF1). This improvement was statistically significant for the experimental group in all sections, but only in the paper folding test section (VT2) for the control group. In the paper folding test, the improvement was similar in both groups. This last result can be justified by the fact that students in the treatment group worked only with two-dimensional models and did not deal without multiple folds which were instead in the test items. The results showed a significant improvement of the experimental group compared to the control group in the Form Board Test (VT1) and in the Hidden Figures Test (CF1). It is possible that the significant improvement of the experimental group on VT1 is linked to the operative apprehensions fostered by paper folding (Duval, 1994). This is in agreement with Kalogirou's et al. (2009) study. The significant improvement in the CF1 results may be due to practice in analysing the geometric configurations required for constructions with tools. Regarding the Surface development Test (VT3), non-significant difference was found between the groups after the treatment, although only the experimental group showed a statistically significant improvement.

From a statistical point of view, the non-significant difference between groups was due to high variabilities in both groups, the high presence of zero scores in both groups (which affects all the measures), and the similar proportion of scores between 0 and 12 in both groups; moreover, the pattern of behaviour of the data in both groups was very similar, except for three cases in the experimental group, whose scores were above 14. At the individual level, within both groups, there were students whose scores improved and those whose scores worsened. In particular, a greater number of students in the experimental group showed improvement compared with the students in the control group. In addition, even though some scores declined, the decrease was less pronounced among participants of the experimental group than among those in the control group. In conclusion, at the global level, the groups had similar performance since the students, as a whole, obtained similar results in the VT3 test. However, when the results were analysed in more detail, a greater tendency towards improvement was found in the experimental group.

The results are in line with other quantitative studies on high school plane geometry that focus on paper folding and DGS separately. In fact, earlier studies have shown that paper folding positively affects high school students' geometric reasoning and visualization skills (Arıcı & Aslan-Tutak, 2015), and that dynamic geometry promotes the raising of high school students' van Hiele levels (Adelabu et al., 2019; Kutluca, 2013). However, the study certainly has some limitations, such as a sample size that is too small to ensure representativeness and generalization of results, and a lack of previous research on the topic. Consequently, further studies involving a larger sample of students are needed. Moreover, it is unfortunate that the study was conducted during the COVID-19 pandemic and that the treatment could only be carried out remotely. This situation prevented students from interacting physically while working with the artefacts, which may have interfered with the effectiveness of the treatment. Therefore, it would be appropriate to replicate the study with in-person treatment to evaluate its efficiency under different conditions.

Recommendations

Despite the difficulty in generalising the results and the necessity of further studies, the findings have shown that the joint use of paper folding and dynamic geometry (in our case GeoGebra) has beneficial effects both in the growth of secondary students' geometric thinking and in the development of their visual-spatial skills. The results of the research led to the formulation of the following recommendations to be considered in geometry teaching. First of all, the regular use of paper folding and GeoGebra is highly recommended from primary school onwards in geometry classes to stimulate the acquisition of in-depth concepts and improve geometric thinking and visuospatial skills. Secondly, given the importance of visuospatial skills in all areas of mathematics, extensive use of paper folding and GeoGebra also contributes to strengthening mathematical thinking. Thirdly, it is necessary to provide adequate training for teachers on the joint use of paper folding and dynamic geometry systems such as GeoGebra.

Acknowledgements

The authors acknowledge the collaboration kindly provided by prof. Teresa Nucci. This work is related to the doctoral dissertation of Giovanna Valori directed by Veronica Albanese and Natividad Adamuz-Povedano. This work was supported by the Ministero dell'Istruzione, dell'Università e della Ricerca (Italy) [grant number Com. Study n. 6064/Liceo Classico Stabili-Trebbiani].

Ethical Issues

The study was carried with permissions of the competent authorities of the school involved and in compliance with current privacy regulations guaranteeing the anonymity of the participants. The participation was voluntary and the informed consent has been requested to participants and parents or legal tutors.

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